THE CAMBRIDGE

MATHEMATICAL JOURNAL.

VOL. III.] NOVEMBER, 1841. [No. XIII.

I.—EXPOSITION OF A GENERAL THEORY OF LINEAR TRANSFORMATIONS. PART I.

By GEORGE BOOLE.

1. THE transformation of homogeneous functions by linear substitutions, is an important and oft-recurring problem of analysis. In the *Mécanique Analytique* of Lagrange, it occupies a very prominent place, and it has been made the subject of a special memoir by Laplace. More recently it has engaged the attention of Lebesgue and Jacobi; the former of whom has extended his investigations to homogeneous functions of the second degree, and of an indefinite number of variables, while the latter has applied the results of such inquiries to the transformation of multiple integrals. A memoir on this subject has also been given to the world by Cauchy; and an ingenious paper by Professor De Morgan, on its geometrical relations, will be found in the 5th volume of the *Cambridge Philosophical Transactions*.

The most general conclusion to which the labours of the above-mentioned writers have led, is, that it is always possible to take away the products of the variables $x_1, x_2, \ldots x_m$, from a proposed homogeneous function of the second degree, Q, by the linear substitution of a new set of variables, $y_1, y_2, \ldots y_m$, connected with the original ones by the relation

$$x_1^2 + x_2^2 \ldots + x_m^2 = y_1^2 + y_2^2 \ldots + y_m^2 \ldots . (1);$$

or in other words, to determine, subject to (1), the values of

the coefficients $A_1, A_2, .. A_m$, in the equation of transformation,

$$Q = A_1 y_1^2 + A_2 y_2^2 \dots + A_m y_m^2 \dots\dots (2).$$

And the method commonly employed in this investigation has been, to substitute, in place of the variables involved in either member of (2), a series of linear functions of the variables involved in the opposite member, to equate coefficients, and to eliminate the unknown constants by aid of the equations of condition similarly obtained from (1). It is in the effecting of this elimination that the principal difficulty of the problem consists; a difficulty arising from the very principle of the method of solution, and therefore not to be evaded; a difficulty moreover so great, that no one has yet shewn how it is to be overcome, when the degree of the function to be transformed rises above the second.

In the above remarks, it is not however intended to convey the idea that this elimination is impossible. Were the final results of the elimination known, it would at once be seen by what combinations of our equations those results might be produced; but this fact brings us no nearer to their actual discovery. Indeed it must on the slightest consideration be manifest, that no such principle of investigation can suffice to the requirements of a problem, which, alike in its primary analysis and in the forms of its final solution, will be shewn to rest on the doctrine of developments, and to involve the processes of the Differential Calculus.

2. The equations (1) and (2) are evidently particular cases of the homogeneous system,

$$h_2(x_1, x_2, \dots x_m) = h'_2(y_1, y_2, \dots y_m) \dots (A_1),$$
$$H_2(x_1, x_2, \dots x_m) = H'_2(y_1, y_2, \dots y_m) \dots (B_1),$$

in which h_2, h'_2, H_2, H'_2, designate homogeneous functions of the second degree; and these again of the more general system,

$$h_n(x_1, x_2, \dots x_m) = h'_n(y_1, y_2, \dots y_m) \dots (A_2),$$
$$H_n(x_1, x_2, \dots x_m) = H'_n(y_1, y_2, \dots y_m) \dots (B_2),$$

h_n, h_n', &c. indicating, in accordance with the above employed notation, homogeneous functions of the n^{th} degree; and the problem in this case mainly refers to the determination of the mutual relations of the constants in (A_2) and (B_2), on the assumption that the second members of those equations are formed respectively from their first, by the same system of linear substitutions. The equations (A_2) and (B_2), I shall,

throughout the remainder of this paper, represent under the abbreviated forms

$$q = r \,..\, (A_3),$$
$$Q = R \,..\, (B_3).$$

Let ν be the number of the terms in the homogeneous function of the n^{th} degree with m variables, and let ι represent in succession each of the series of integers from $m+1$ to ν inclusive; also, let $\alpha, \beta, \gamma \ldots \mu$ be indefinite integers, varying from 0 to $n-1$ inclusive, and entering into every possible combination subject to the condition of homogeneity,

$$\alpha + \beta + \gamma \ldots + \mu = n;$$

then may the functions q, r, Q, R, be represented under the general forms

$$\left.\begin{aligned} q &= a_1 x_1^n + a_2 x_2^n \ldots + a_m x_m^n + \Sigma a_\iota x_1^\alpha x_2^\beta \ldots x_m^\mu \\ r &= b_1 y_1^n + b_2 y_2^n \ldots + b_m y_m^n + \Sigma b_\iota y_1^\alpha y_2^\beta \ldots y_m^\mu \\ Q &= A_1 x_1^n + A_2 x_2^n \ldots + A_m x_m^n + \Sigma A_\iota x_1^\alpha x_2^\beta \ldots x_m^\mu \\ R &= B_1 y_1^n + B_2 y_2^n \ldots + B_m y_m^n + \Sigma B_\iota y_1^\alpha y_2^\beta \ldots y_m^\mu \end{aligned}\right\} \ldots (3),$$

and our primary object, as above observed, is to determine the relations by which the 4ν quantities $a_1, a_2 \ldots a_\nu, \ldots b_1, b_2 \ldots b_\nu$, $A_1, A_2 \ldots A_\nu$, $B_1, B_2 \ldots B^\nu$, are held in mutual dependence.

By Taylor's theorem, it is evident that the equations (A_3), (B_3), cannot exist in their utmost generality, unless the various orders of equations thence obtained by successive differentiation, are also satisfied. Confining our attention to those of the first order, and differentiating relatively to $y_1, y_2 \ldots y_m$, we have, from (B_3),

$$\left.\begin{aligned} \frac{dQ}{dx_1}\frac{dx_1}{dy_1} + \frac{dQ}{dx_2}\frac{dx_2}{dy_1} \ldots + \frac{dQ}{dx_m}\frac{dx_m}{dy_1} &= \frac{dR}{dy_1} \\ \frac{dQ}{dx_1}\frac{dx_1}{dy_2} + \frac{dQ}{dx_2}\frac{dx_2}{dy_2} \ldots + \frac{dQ}{dx_m}\frac{dx_m}{dy_2} &= \frac{dR}{dy_2} \\ \cdots\cdots\cdots\cdots & \\ \frac{dQ}{dx_1}\frac{dx_1}{dy_m} + \frac{dQ}{dx_2}\frac{dx_2}{dy_m} \ldots + \frac{dQ}{dx_m}\frac{dx_m}{dy_m} &= \frac{dR}{dy_m} \end{aligned}\right\} \ldots (4).$$

3. Let us suppose the actual linear relations among the two sets of variables to be represented under the forms,

$$\left.\begin{aligned} x_1 &= \lambda_1 y_1 + \lambda_2 y_2 \ldots + \lambda_m y_m \\ x_2 &= \mu_1 y_1 + \mu_2 y_2 \ldots + \mu_m y_m \\ &\cdots\cdots \\ x_m &= \rho_1 y_1 + \rho_2 y_2 \ldots + \rho_m y_m \end{aligned}\right\} \ldots (5).$$

Then do the above equations (4) become

$$\left.\begin{array}{l}
\lambda_1 \dfrac{dQ}{dx_1} + \mu_1 \dfrac{dQ}{dx_2} \ldots + \rho_1 \dfrac{dQ}{dx_m} = \dfrac{dR}{dy_1} \\
\lambda_2 \dfrac{dQ}{dx_1} + \mu_2 \dfrac{dQ}{dx_2} \ldots + \rho_2 \dfrac{dQ}{dx_m} = \dfrac{dR}{dy_2} \\
\quad . \qquad . \qquad . \qquad . \qquad . \qquad . \\
\lambda_m \dfrac{dQ}{dx_1} + \mu_m \dfrac{dQ}{dx_2} \ldots + \rho_m \dfrac{dQ}{dx_m} = \dfrac{dR}{dy_m}
\end{array}\right\} \ldots (6).$$

We suppose the coefficients $\lambda_1, \lambda_2, \ldots \lambda_m, \mu_1, \mu_2 \ldots \mu_m$, &c. in the linear theorems (5) to be of finite value; and on this supposition it is clear that the assumption in (6) of the simultaneous conditions,

$$\frac{dQ}{dx_1} = 0, \quad \frac{dQ}{dx_2} = 0, \ldots \frac{dQ}{dx_m} = 0 \ldots . (7)$$

will induce, as a necessary consequence, the fulfilment of the simultaneous and similar conditions

$$\frac{dR}{dy_1} = 0, \quad \frac{dR}{dy_2} = 0, \ldots \frac{dR}{dy_m} = 0 \ldots . (8).$$

The converse of this proposition is not so universally true.

If we suppose the second members of (6) to vanish, and linearly eliminate $\dfrac{dQ}{dx_1}, \dfrac{dQ}{dx_2}, \ldots \dfrac{dQ}{dx_m}$, from the first members thus equated to 0, we shall obtain a final equation among the constants,

$$F(\lambda_1, \mu_1, \ldots \rho_1, \ldots \lambda_m, \mu_m, \ldots \rho_m) = 0 \ldots . (9),$$

which, if satisfied, will indicate, that the proposed conditions, (8), may coexist, without the simultaneous evanescence of $\dfrac{dQ}{dx_1}, \dfrac{dQ}{dx_2}, \ldots \dfrac{dQ}{dx_m}$. These circumstances are here noticed, because they will be found to reappear, as the cause of certain peculiarities in the final solution. On the nature and meaning of the condition (9), it will for the present be sufficient to remark, that it analytically corresponds to those cases in which, while definite values are attributed to the one set of variables $y_1, y_2, \ldots y_m$, those of the other set $x_1, x_2, \ldots x_m$, become arbitrary or infinite, and that its geometrical interpretation has reference to certain cases of impossible transformation, such as, for example, the change of co-ordinates from a pair of axes having a given inclination to another pair mutually coinciding. Omitting therefore the further

consideration of this case of failure, we proceed to examine the conclusions which may be drawn from the otherwise dependent, and always compatible systems (7) and (8).

4. Employing the value of Q given in (3), the system (7) becomes

$$\left.\begin{array}{l} nA_1x_1^{n-1} + \Sigma A_\iota a x_1^{a-1}x_2^{\beta} \ldots x_m^{\mu} = 0 \\ nA_2x_2^{n-1} + \Sigma A_\iota \beta x_1^{a}x_2^{\beta-1} \ldots x_m^{\mu} = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ nA_mx_m^{n-1} + \Sigma A_\iota \mu x_1^{a}x_2^{\beta} \ldots x_m^{\mu-1} = 0 \end{array}\right\} \ldots (10).$$

These equations are homogeneous and of the $(n-1)^{\text{th}}$ degree: we are now to inquire under what conditions they may be satisfied. The first which presents itself is found in the supposition, that $x_1, x_2, \ldots x_m$, simultaneously vanish; but this leads to no ulterior result. The only remaining one will be determined by examining the result arising from the elimination of the variables. To this object we therefore now direct our attention.

If to any two equations of the n^{th} degree,

$$ax^n + bx^{n-1} + cx^{n-2} \ldots + k = 0 \ldots . (11),$$

$$a'x^n + b'x^{n-1} + cx'^{n-2} \ldots + k' = 0 \ldots . (12),$$

we apply the process employed in finding a common measure, and equate the successive remainders to 0, which in such case we evidently may do, we shall in the first stage of the process obtain an equation of the $(n-1)^{\text{th}}$ degree, in the next stage we shall have an equation of the $(n-2)^{\text{th}}$ degree; and thus at each successive stage of the process depressing the degree of our resulting equation by unity, shall finally arrive at an equation involving only the coefficients a, b, a', b', &c. with the constants k and k', and which may be regarded as the result of the final elimination of x from (11) and (12).

By applying this method to equations (1) and (2) of (10), we shall be able to eliminate x_1, and repeating the process successively on the independent pairs (2) and (3), (3) and (4), $(m-1)$ and (m), of the same system, shall form a system of $m-1$ equations, involving only the variables $x_2, x_3, \ldots x_m$, and from which x_1 will have entirely disappeared. These equations may be treated in the same way as those of (10), and will lead by the elimination of x_2, to a system of $m-2$ equations involving only the variables, $x_3, x_4, \ldots x_m$. Thus by successive stages shall we obtain from the m equations (10) with m variables,

$m-1$ equations involving $m-1$ variables,
$m-2$ equations . . . . $m-2$ variables,
. . . .
2 equations . . . . 2 variables,
1 final equation . . . . constants;

since all these equations are, like the original ones, homogeneous, and from the last the variable will divide out. It is also to be observed, that the two equations given in the last step but one of the process of reduction, will be linear, as will be more distinctly seen in the examples given below. This circumstance is important, because it restricts the general solution to the condition of linearity among the variables; the reason will appear in the sequel.

The last obtained of the above column of results, we shall designate by the symbol θ, applied to the original function Q. Thus, if, to adopt the common notation, Q were a homogeneous function of the second degree, of the form,

$$Q = Ax^2 + 2Bxy + Cy^2 \ldots (13);$$

then on eliminating from the derived equations,

$$Ax + By = 0,$$
$$Bx + Cy = 0,$$

(for $\dfrac{dQ}{dx} = 2Ax + 2By$, and $\dfrac{dQ}{dy} = 2Bx + 2Cy$), we find

$$\theta(Q) = B^2 - AC, \text{ or } AC - B^2 \ldots (14).$$

Again, if we have

$$Q = Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy \ldots (15),$$

the equations between which the elimination is to be effected will be

$$\left.\begin{aligned} Ax + Fy + Ez &= 0 \\ Fx + By + Dz &= 0 \\ Ex + Dy + Cz &= 0 \end{aligned}\right\} \ldots (16),$$

whence the result sought becomes, on reduction,

$$\theta(Q) = ABC + 2DEF - (AD^2 + BE^2 + CF^2) \ldots (17).$$

Finally, if Q be of the form,

$$Q = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 \ldots (18),$$

we shall, on taking the first differential coefficients, have

$$Ax^2 + 2Bxy + Cy^2 = 0,$$
$$Bx^2 + 2Cxy + Dy^2 = 0.$$

From these, eliminating x^2 and y^2, and dividing the results by y and by x respectively, we find

$$\left.\begin{array}{l} 2(B^2 - AC)\,x - (AD - BC)\,y = 0 \\ (AD - BC)\,x - 2(C^2 - BD)\;y = 0 \end{array}\right\} \ldots\ldots (19),$$

which, in accordance with a previous remark, are linear. Hence, on elimination,

$$\theta(Q) = (AD - BC)^2 - 4(B^2 - AC)(C^2 - BD) \ldots\ldots (20);$$

and in a similar way may we proceed for more complicated cases.

5. It is evident, to resume our original notation, that the equation, $\theta(Q) = 0$, expresses the relation which must be fulfilled among the constants $A_1, A_2, \ldots A_m$, in order that the equations, (7), may admit of being satisfied without the simultaneous vanishing of $x_1, x_2, \ldots x_m$. We shall suppose this condition to be fulfilled, and that the equations, (7), are in reality satisfied, while $x_1, x_2, \ldots x_m$ retain actual values. By the reasoning of sect. 3, it has been shewn that the equations, (8), will also be satisfied, and by inspection of the linear theorems (5), it will be seen, that $y_1, y_2 \ldots y_m$ cannot all simultaneously vanish, consistently with our assumptions relatively to $x_1, x_2, \ldots x_m$; hence it will be necessary that the condition $\theta(R) = 0$, be also satisfied among the constants $B_1, B_2, \ldots B_m$. Thus the condition $\theta(Q) = 0$, when satisfied, involves as a necessary consequence, the fulfilment of the analogous condition, $\theta(R) = 0$; and the same relation of mutual dependence exists between $\theta(q)$ and $\theta(r)$.

Now it is scarcely necessary to observe, that the constitution of the two functions, Q and q, relatively to the constants they involve, will not generally be such, as that the conditions, $\theta(Q) = 0$, $\theta(q) = 0$, shall be thereby satisfied. Here therefore we are to inquire, whether it is not possible to obtain from these a third function, which, by the particular determinations of an arbitrary constant, shall enable us to satisfy the conditions required. In order to effect this, add the primitive equations (A_3), (B_3), after having multiplied the former by an indeterminate constant quantity h, we have

$$Q + hq = R + hr \ldots\ldots (21).$$

Like each of the original equations, this will be homogeneous. Considered as the subject of the argument above developed, it leads to the conclusion, that the two systems of equations,

$$\frac{d(Q+hq)}{dx_1}=0, \quad \frac{d(Q+hq)}{dx_2}=0, \ldots \frac{d(Q+hq)}{dx_m}=0 \ldots (22),$$

$$\frac{d(R+hr)}{dy_1}=0, \quad \frac{d(R+hr)}{dy_2}=0, \ldots \frac{d(R+hr)}{dy_m}=0 \ldots (23),$$

are mutually dependent; and as a further consequence of the same process of reasoning, that if the constant h be so determined as to satisfy the equation,

$$\theta(Q+hq)=0 \ldots (24),$$

then will the analogous equation

$$\theta(R+hr)=0 \ldots (25)$$

be satisfied also. I shall now shew that this principle involves the complete solution of the problem under consideration.

6. Let ϕ, as a symbol of functionality, indicate the combinations of the constants in $\theta(Q)$, $\theta(R)$, &c., so that

$$\theta(Q)=\phi(A_1 A_2, \ldots A_m), \quad \theta(R)=\phi(B_1 B_2, \ldots B_m),$$
$$\theta(q)=\phi(a_1 a_2, \ldots a_m), \quad \theta(r)=\phi(b_1 b_2, \ldots b_m).$$

Then substituting in (21) the forms assigned in (3), we have

$$(A_1+ha_1)x_1^n+(A_2+ha_2)x_2^n \ldots +(A_m+ha_m)x_m^n+\Sigma(A_\iota+ha_\iota)x_1^\alpha x_2^\beta \ldots x_m^\mu$$
$$=(B_1+hb_1)y_1^n+(B_2+hb_2)y_2^n \ldots +(B_m+hb_m)y_m^n+\Sigma(B_\iota+hb_\iota)y_1^\alpha y_2^\beta \ldots y_m^\mu$$
$$\ldots (26).$$

And the equations (24), (25), on replacing θ by ϕ, thus give

$$\phi\{(A_1+ha_1), \quad (A_2+ha_2), \ldots (A_\nu+ha_\nu)\}=0 \ldots (27),$$
$$\phi\{(B_1+hb_1), \quad (B_2+hb_2), \ldots (B_\nu+hb_\nu)\}=0 \ldots (28).$$

As the values of h satisfying these equations must be identical, and as those values are to be sought by the development of their first members in ascending or descending powers of that quantity, it is evident that those equations, in their developed forms, must be equivalent relatively to that quantity. If we then observe that the terms independent of h in the two developments, are $\phi(A_1, A_2 \ldots A_\nu)$ and $\phi(B_1, B_2 \ldots B_\nu)$ respectively, and that the corresponding coefficients of the highest power of h are $\phi(a_1, a_2 \ldots a_\nu)$ and $\phi(b_1, b_2 \ldots b_\nu)$ respectively, and that the intermediate terms are formed according to the ordinary laws of development by Taylor's theorem, it will be manifest, that in order to establish the proposed equivalence, we must have

$$\frac{\phi(A_1, A_2 \ldots A_\nu)}{\phi(a_1, a_2 \ldots a_\nu)}=\frac{\phi(B_1, B_2 \ldots B_\nu)}{\phi(b_1, b_2 \ldots b_\nu)} \ldots (29),$$

$$\frac{\left(a_1\frac{d}{dA_1}+a_2\frac{d}{dA_2}\,..\,+a_\nu\frac{d}{dA_\nu}\right)\phi\left(A_1, A_2\,..\,A_\nu\right)}{\phi\left(a_1\, a_2\,..\,a_\nu\right)}$$
$$=\frac{\left(b_1\frac{d}{dB_1}+b_2\frac{d}{dB_2}\,..\,+b_\nu\frac{d}{dB_\nu}\right)\phi\left(B_1\, B_2\,..\,B_\nu\right)}{\phi\left(b_1\, b_2\,..\,b_\nu\right)}\,..\,(30),$$

$$\frac{\left(a_1\frac{d}{dA_1}+a_2\frac{d}{dA_2}\,..\,+a_\nu\frac{d}{dA_\nu}\right)^\gamma\phi\left(A_1, A_2\,..\,A_\nu\right)}{\phi\left(a_1\, a_2\,..\,a_\nu\right)}$$
$$=\frac{\left(b_1\frac{d}{dB_1}+b_2\frac{d}{dB_2}\,..\,+b_\nu\frac{d}{dB_\nu}\right)^\gamma\phi\left(B_1, B_2\,..\,B_\nu\right)}{\phi\left(b_1\, b_2\,..\,b_\nu\right)}\,....(31),$$

if we represent by γ the degree of $\phi\left(A_1, A_2, ..\, A_\nu\right)$, which will of course determine the limit of the orders of differentiation. On replacing $\phi\left(A_1, A_2, ..\, A_\nu\right)$ by $\theta\left(Q\right)$, &c., our equations become

$$\frac{\theta\left(Q\right)}{\theta\left(q\right)}=\frac{\theta\left(R\right)}{\theta\left(r\right)}\,......\,(32,)$$

$$\frac{\left(a_1\frac{d}{dA_1}+a_2\frac{d}{dA_2}\,..\,+a_\nu\frac{d}{dA_\nu}\right)\theta\left(Q\right)}{\theta\left(q\right)}=\frac{\left(b_1\frac{d}{dB_1}+b_2\frac{d}{dB_2}\,..\,+b_\nu\frac{d}{dB_\nu}\right)\theta\left(R\right)}{\theta\left(r\right)},$$
$$......(33),$$

$$\frac{\left(a_1\frac{d}{dA_1}+a_2\frac{d}{dA_2}\,..\,+a_\nu\frac{d}{dA_\nu}\right)^\gamma\theta\left(Q\right)}{\theta\left(q\right)}=\frac{\left(b_1\frac{d}{dB_1}+b_2\frac{d}{dB_2}\,..\,+b_\nu\frac{d}{dB_\nu}\right)^\gamma\theta\left(R\right)}{\theta\left(r\right)}$$
$$......(34),$$

which are the simplest forms under which the full solution can be placed, and are a direct consequence of the relations (24) and (25). The results of this part of our inquiry may therefore be comprised in the following general proposition.

A. *If* Q *and* q *represent two similar homogeneous functions of the same degree, which are linearly and similarly transformed into* R *and* r *respectively, and if the symbol* θ *before a proposed homogeneous function be understood to indicate the result of the elimination of the variables, from the first differential coefficients of that function, equated to* 0, *then are the relations among the coefficients of* Q, R, q, r, *the same as are necessary in order to verify the assumed identity of the equations,*

$$\theta\left(Q+hq\right)=0,\quad \theta\left(R+hr\right)=0,$$

relatively to the constant h.

It may be proper to observe, that the analysis on which the above theorem is here made to depend, is considerably different from that by which I originally obtained it. This, in fact, consisted in an extension of the method which I on a former occasion employed, when treating the same subject in the pages of this Journal, vide No. VIII. Vol. II. p. 64.

7. Besides the literal coefficients $A_1, A_2, \ldots A_\nu, B_1, B_2$, &c., it commonly happens, that the functions Q, q, R, r, as in the examples of section 4, are in some of their terms affected with numerical multipliers. Provided however that these multipliers are the same, and are similarly employed, in Q and q, and again in R and r, no change will be thereby introduced in the symbolical forms of the general solution, (32), (33), (34). For let $k_1, k_2 \ldots k_\nu$ be numerical quantities, and let the compound coefficients of the several terms in Q be $k_1 A_1$, $k_2 A_2, \ldots k_\nu A_\nu$, and of those in q taken in the same order $k_1 a_1$, $k_2 a_2, \ldots k_\nu a_\nu$, then will those of $Q + hq$ be

$$k_1 (A_1 + ha_1),\ k_2 (A_2 + ha_2) \ldots k_\nu (A_\nu + ha_\nu) \ldots\ldots (35).$$

Now $k_1, k_2, \ldots k_\nu$, being numerical, will not distinctively appear in $\theta(Q)$ and $\theta(q)$ which will therefore, as before, assume the form

$$\phi(A_1 A_2 \ldots A_\nu), \quad \phi(a_1 a_2 \ldots a_\nu).$$

Hence also will $\theta(Q + hq)$ become

$$\theta(Q + hq) = \phi\{(A_1 + ha_1)(A_2 + ha_2) \ldots (A_\nu + ha_\nu)\} \ldots\ldots (36),$$

and adopting the same class of numerical coefficients with B_1, $B_2, \ldots B_\nu$, in the expression of R and r,

$$\theta(R + hr) = \phi\{(B_1 + hb_1)(B_2 + hb_2) \ldots (B_\nu + hb_\nu)\} \ldots\ldots (37).$$

These forms are identical with those employed in the previous sections, although the interpretation of ϕ will be modified by the introduction of $k_1, k_2, \ldots k_\nu$. Taken in connexion with the theorem A, they lead to the same series of equations, presented under the general type,

$$\frac{\left(a_1 \dfrac{d}{dA_1} + a_2 \dfrac{d}{dA_2} \ldots + a_\nu \dfrac{d}{dA_\nu}\right)^\eta \theta(Q)}{\theta(q)} = \frac{\left(b_1 \dfrac{d}{dB_1} + b_2 \dfrac{d}{dB_2} \ldots + b_\nu \dfrac{d}{dB_\nu}\right)^\eta \theta(R)}{\theta(r)} \ldots\ldots\ldots (38),$$

η being an indefinite integer varying from 0 upwards.

If $\eta = \gamma$ (which is the highest value it can receive without causing both sides to vanish), it is evident, by the nature of developments, that this equation will become

$$\frac{\theta(q)}{\theta(q)} = \frac{\theta(r)}{\theta(r)}, \text{ or } 1 = 1.$$

Hence the values of η which are to be employed, range from 0 to $\gamma - 1$. Of the series of conditions (32), (33), (34), the uppermost is peculiarly deserving of attention, and it may be shewn that the succeeding ones might be formed, by the extension of A to the whole class of homogeneous equations represented under the type,

$$Q + nq = R + nr,$$

n being entirely arbitrary. It is in fact on inspection evident, that the series of conditions found by the development of both members of the equation

$$\frac{\theta(Q + nq)}{\theta(q)} = \frac{\theta(R + nr)}{\theta(r)}$$

would reproduce the system (32), (33), (34).

8. As a first example of the application of the above theorems, I select the very simple case

$$ax^2 + 2bxy + cy^2 = a'x'^2 + 2b'x'y' + c'y'^2 \ldots\ldots (39),$$

$$Ax^2 + 2Bxy + Cy^2 = A'x'^2 + 2B'x'y' + C'y'^2 \ldots. (40).$$

Here by (14) $\theta(Q) = AC - B^2$, $\theta(q) = ac - b^2$, $\theta(R) = A'C' - B'^2$, $\theta(r) = a'c' - b'^2$, which values are to be employed in the symbolical forms of the general solution (38) as applied to this particular case, viz.

$$\frac{\theta(Q)}{\theta(q)} = \frac{\theta(R)}{\theta(r)}$$

$$\frac{\left(a\dfrac{d}{dA} + b\dfrac{d}{dB} + c\dfrac{d}{dC}\right)\theta(Q)}{\theta(q)} = \frac{\left(a'\dfrac{d}{dA'} + b'\dfrac{d}{dB'} + c'\dfrac{d}{dC'}\right)\theta(R)}{\theta(r)}.$$

This being done, and the requisite differentiations effected, we find

$$\frac{AC - B^2}{ac - b^2} = \frac{A'C' - B'^2}{a'c' - b'^2} \ldots\ldots.. (41),$$

$$\frac{aC - 2bB + cA}{ac - b^2} = \frac{a'C' - 2b'B' + c'A'}{a'c' - b'^2} \ldots (42).$$

The equation (39) may be regarded as characteristic of the nature of the transformation to be effected. Should that correspond to the geometrical idea of a change of co-ordinates, from a pair of axes x, y, making an angle θ, to another pair x', y', whose angle of inclination is θ', then will (39) become

$$x^2 + 2xy\cos\theta + y^2 = x'^2 + 2x'y'\cos\theta' + y'^2$$

$$\therefore\ a = 1,\ b = \cos\theta,\ c = 1,\quad a' = 1,\ b' = \cos\theta',\ c' = 1.$$

Hence (41) and (42) give

$$\frac{AC-B^2}{(\sin\theta)^2}=\frac{A'C'-B'^2}{(\sin\theta')^2}\;\ldots\ldots\ldots(43),$$

$$\frac{A-2B\cos\theta+C}{(\sin\theta)^2}=\frac{A'-2B'\cos\theta'+C'}{(\sin\theta')^2}\;\ldots(44).$$

We pass next to the important case

$$\left.\begin{array}{l} x^2+y^2+z^2=x'^2+y'^2+z'^2, \\ Ax^2+By^2+Cz^2+2Dyz+2Exz+2Fxy \\ \quad =A'x'^2+B'y'^2+C'z'^2+2D'y'z'+2E'x'z'+2F'x'y' \end{array}\right\}\ldots(45).$$

Here, by reference to (17), we find

$$\theta(Q)=ABC+2DEF-(AD^2+BE^2+CF^2).\,\theta(q)=1.\ldots(46),$$

$$\theta(R)=AB'C'+2D'E'F'-(A'D'^2+B'E'^2+C'F'^2).\,\theta(r)=1.\,.(47),$$

which expressions are to be employed in the symbolical forms,

$$\frac{\theta(Q)}{\theta(q)}=\frac{\theta(R)}{\theta(r)},$$

$$\frac{\left(\frac{d}{dA}+\frac{d}{dB}+\frac{d}{dC}\right)\theta(Q)}{\theta(q)}=\frac{\left(\frac{d}{dA'}+\frac{d}{dB'}+\frac{d}{dC'}\right)\theta R}{\theta(r)},$$

$$\frac{\left(\frac{d}{dA}+\frac{d}{dB}+\frac{d}{dC}\right)^2\theta(Q)}{\theta(q)}=\frac{\left(\frac{d}{dA'}+\frac{d}{dB'}+\frac{d}{dC'}\right)^2\theta(R)}{\theta(r)},$$

whence, by mere inspection, we have

$$ABC+2DEF-(AD^2+BE^2+CF^2)$$
$$=A'B'C'+2D'E'F'-(A'D'^2+B'E'^2+C'F'^2).\ldots(48),$$

$$AB+BC+AC-(D^2+E^2+F^2)=A'B'+B'C'+A'C'-(D'^2+E'^2+F'^2)(49),$$

$$A+B+C=A'+B'+C'\;\ldots\ldots\ldots\ldots\ldots(50).$$

If D' E' F' are supposed to vanish, the above system becomes equivalent to the remarkable cubic, so frequently met with in Analytical Mechanics, and the Geometry of Space.

Were the number of the variables four, the conditions of the problem remaining in other respects unaltered, we should in the same way obtain an equation of the fourth degree, or rather a system of equations thereto equivalent, determining the values of the four constants in R, and so on to any proposed number of variables.

9. Let us now attribute to our primitive equations the more general forms,

$$\left.\begin{array}{l} ax^2+by^2+cz^2+2dyz+2exz+2fxy \\ =a'x'^2+b'y'^2+c'z'^2+2d'y'z'+2e'x'z'+2f'x'y' \end{array}\right\}\text{for } q=r\ldots(51),$$

$$\left.\begin{aligned} &Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy \\ &= A'x'^2 + B'y'^2 + C'z'^2 + 2D'y'z' + 2E'x'y' + 2F'x'y' \end{aligned}\right\} \text{for } Q = R.. (52).$$

Here to the values of $\theta(Q)$ and $\theta(R)$ as given in the last example, we must add

$$\theta(q) = abc + 2def - (ad^2 + be^2 + cf^2).\ \theta(r) = a'b'c' + 2d'e'f' - (a'd'^2 + b'e'^2 + c'f'^2).$$

The symbolical forms of the solution for this case, are at once seen to be

$$\frac{\theta(Q)}{\theta(q)} = \frac{\theta(R)}{\theta(r)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (53),$$

$$\frac{\left(a\frac{d}{dA} + b\frac{d}{dB} \ldots + f\frac{d}{dF}\right)\theta(Q)}{\theta(q)} = \frac{\left(a'\frac{d}{dA'} + b'\frac{d}{dB'} \ldots + f'\frac{d}{dF'}\right)\theta(R)}{\theta(r)} \ldots (54)$$

$$\frac{\left(a\frac{d}{dA} + b\frac{d}{dB} \ldots + f\frac{d}{dF}\right)^2\theta(Q)}{\theta(q)} = \frac{\left(a'\frac{d}{dA'} + b'\frac{d}{dB'} + f'\frac{d}{dF'}\right)^2\theta(R)}{\theta(r)} \ldots (55).$$

When the algebraic development of (54) is determined, that of (55) will be found, by simply changing in the former $a, b, c,$ &c., into $A, B, C,$ &c., and *vice-versa*. In exhibiting the results of these developments, it will be convenient to represent the differential coefficients of $\theta(Q)$, $\theta(R)$ &c., by subsidiary quantities. Assume therefore

$$\begin{array}{lll} L = BC - D^2, & M = AC - E^2, & N = AB - F^2), \\ S = 2(EF - AD), & T = 2(DF - BC), & U = 2(DE - CF), \\ L' = B'C' - D'^2, \&c., & l = bc - d^2, \&c., & l' = b'c' - d'^2, \&c. \end{array}$$

Then will (53), (54), (55), give, on effecting the operations indicated,

$$\frac{ABC + 2DEF - (AD^2 + BE^2 + CF^2)}{abc + 2def - (ad^2 + be^2 + cf^2)} = \frac{A'B'C' + 2D'E'F' - (A'D'^2 + B'E'^2 + C'F'^2)}{a'b'c' + 2d'e'f' - (a'd'^2 + b'e'^2 + c'f'^2)} \ldots (56),$$

$$\frac{aL + bM + cN + dS + eT + fU}{abc + 2def - (ad^2 + be^2 + cf^2)} = \frac{a'L' + b'M' + c'N' + d'S' + e'T' + f'U'}{a'b'c' + 2d'e'f' - (a'd'^2 + b'e'^2 + c'f'^2)} \ldots (57),$$

$$\frac{Al + Bm + Cn + Ds + Et + Fu}{abc + 2def - (ad^2 + be^2 + cf^2)} = \frac{A'l' + B'm' + C'n' + D's' + E't' + F'u'}{a'b'c' + 2d'e'f' - (a'd'^2 + b'e'^2 + c'f'^2)} \ldots (58).$$

If we wish by the above transformation to represent a change of co-ordinates, from axes x, y, z, given in relative position by the equations $\cos yz = \cos\phi$, $\cos xz = \cos\psi$, $\cos\ xy = \cos\chi$, to another system of axes x', y', z', whose inclinations are similarly determined by the angles ϕ', ψ', χ', then will our first equation (51) become

$$\left.\begin{array}{l} x^2 + y^2 + z^2 + 2yz\cos\phi + 2xz\cos\psi + 2xy\cos\chi \\ = x'^2 + y'^2 + z'^2 + 2y'z'\cos\phi' + 2x'z'\cos\psi' + 2x'y'\cos\chi' \end{array}\right\} \ldots (59),$$

so that it will only be necessary in the formulæ of solution (56), (57), (58), to make

$$\begin{array}{l} a = b = c = 1,\ d = \cos\phi,\quad e = \cos\psi,\ f = \cos\chi, \\ a' = b' = c' = 1,\ d' = \cos\phi',\quad e' = \cos\psi',\ f' = \cos\chi', \\ l = (\sin\phi)^2,\quad m = (\sin\psi)^2,\ n = (\sin\chi)^2,\ \&c. \end{array}$$

in order to obtain the relations sought.

10. In further illustration of the general method, let us now take an example of the transformation of homogeneous functions of the third degree, our primitive equations being placed under the forms,

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = a'x'^3 + 3b'x'^2y' + 3c'x'y'^2 + d'y'^3 \ldots (60),$$

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 = A'x'^3 + 3B'x'^2y' + 3C'x'y'^2 + D'y'^3 \ldots (61).$$

Here by (19) we have

$$\theta(Q) = (AD - BC)^2 - 4(B^2 - AC)(C^2 - BD).$$

$$\theta(q) = (ad - bc)^2 - 4(b^2 - ac)(c^2 - bd),$$

$$\theta(R) = (A'D' - B'C')^2 - 4(B'^2 - A'C')(C'^2 - B'D').$$

$$\theta(r) = (a'd' - b'c')^2 - 4(b'^2 - a'c')(c'^2 - b'd'),$$

which, as in former cases, are to be substituted in the general symbolical forms,

$$\frac{\theta(Q)}{\theta(q)} = \frac{\theta(R)}{\theta(r)} \ldots (62),$$

$$\frac{\left(a\dfrac{d}{dA} + b\dfrac{d}{dB} + c\dfrac{d}{dC} + d\dfrac{d}{dD}\right)\theta(Q)}{\theta(q)} = \frac{\left(a'\dfrac{d}{dA'} + b'\dfrac{d}{dB'} + c'\dfrac{d}{dC'} + d'\dfrac{d}{dD'}\right)\theta(R)}{\theta(r)} \ldots (63),$$

$$\frac{\left(a\dfrac{d}{dA} + b\dfrac{d}{dB} + c\dfrac{d}{dC} + d\dfrac{d}{dD}\right)^3\theta(Q)}{\theta(q)} = \frac{\left(a'\dfrac{d}{dA'} + b'\dfrac{d}{dB'} + c'\dfrac{d}{dC'} + d'\dfrac{d}{dD'}\right)^3\theta(R)}{\theta(r)} \ldots (64).$$

The requisite differentiations being performed, we shall, as in the examples already given, be in possession of the final algebraic relations among the constants of (60) and (61);

relations which, in this case, will evidently be of a somewhat complicated character.

Instead however of employing the above method, we may, by the direct application of our fundamental proposition, A, demonstrated in §. 6, obtain at once a result conveniently adapted for numerical computation. For this purpose, having multiplied the upper equation (60) by h, and to the result added the lower (61), let

$$A + ha = A_1,\ B + hb = B_1, \&c.,\ A' + ha' = A'_1, \&c. \ldots. (65),$$

so that the resulting equation, $Q + hq = \boldsymbol{R} + hr$, may assume the form

$$A_1x^3+3B_1x^2y+3C_1xy^2+D_1y^3=A_1'x'^3+3B_1'x'^2y'+3C_1'x'y'^2+D_1'y'^3 \ldots (66).$$

Then, by the proposition in question, the two final equations

$$(A_1D_1 - B_1C_1)^2 - 4(B_1^2 - A_1C_1)(C_1^2 - B_1D_1) = 0 \ldots (67),$$

$$(A_1'D_1' - B_1'C_1')^2 - 4(B_1'^2 - A_1'C_1')(C_1'^2 - B_1'D_1') = 0 \ldots (68),$$

must be identical relatively to h.

Suppose, for example, it were required to determine whether the equations

$$x^3 - y^3 = 3x'^2y' + 3x'y'^2 + y'^3 \ldots . (69),$$

$$x^3 - xy^2 = 2x'^2y' + 3x'y'^2 + y'^3 \ldots . (70),$$

are derivable from a common system of linear relations, connecting x and y with x' and y'. Here $(69) \times h + (70)$ gives

$$(1 + h)x^3 - xy^2 - hy^3 = (2 + 3h)x'^2y' + 3(1 + h)x'y'^2 + (1 + h)y'^3,$$

whence, by comparison with (66),

$$A_1 = 1 + h.\ B_1 = 0.\ C_1 = -\tfrac{1}{3}.\ D_1 = -h\ \ A_1' = 0.\ B_1' = \frac{2 + 3h}{3}\ \ C_1' = D_1' = 1 + h.$$

The substitution of the above values of A_1, B_1, C_1, D_1, in (67), gives

$$h^4 + 2h^3 + h^2 - \frac{4}{27}h - \frac{4}{27} = 0 \ldots . (71).$$

Again, substituting for A_1', B_1', C_1', D_1', in (68), we obtain on reduction the very same equation, viz.

$$h^4 + 2h^3 + h^2 - \frac{4}{27}h - \frac{4}{27} = 0 \ldots . (72),$$

and from the identity of these results infer, that our primitive equations, (69) and (70), are in reality derived from a common system of linear transformations.

11. The determination of the actual values of the constants involved in the linear theorems, connecting the two sets of variables, constitutes a separate branch of our general investiga-

tions. In proceeding to the discussion of this part of the subject, it will be necessary to resume the notation adopted in the first sections of this memoir.

On referring to §. 4, the reader will perceive, that in the last stage but one of the process of elimination, by which we arrive at $\theta(Q)$, we pass through two equations involving two variables. Those equations, it is there observed, and the truth of the remark might easily be proved, are linear. Applying this observation to the process by which $\theta(Q + hq)$ might be similarly obtained, we see, that in whatever order the elimination is effected, we, in the last stage but one, meet with two linear equations, involving the two variables which are last eliminated. Of these equations, it is however clear that one only can be independent, in consequence of the relation $\theta(Q + hq) = 0$, which we here suppose to be satisfied. Now as the order in which the variables are eliminated is indifferent, so that any two of them may be left as the subjects for the linear relations above mentioned, it is evident that in the whole there must exist $m - 1$ such relations, connecting linearly, and by independent pairs, the m variables, $x_1, x_2, . . x_m$. These relations may be put under the form

$$\frac{x_1}{l_1} = \frac{x_2}{l_2} . . = \frac{x_m}{l_m} . . . . (73),$$

$l_1, l_2, . . l_m$, being functions of the coefficients of the several terms in $(Q + hq)$. Thus may the m independent equations

$$\frac{d(Q + hq)}{dx_1} = 0, \quad \frac{d(Q + hq)}{dx_2} = 0 . ., \quad \frac{d(Q + hq)}{dx_m} = 0 . . . . (74),$$

be considered as having merged into the m independent equation

$$\frac{x_1}{l_1} = \frac{x_2}{l_2} . . = \frac{x_m}{l_m}, \quad \theta(Q + hq) = 0 \quad . . . . (75).$$

Similarly, in the process of elimination, may the m independent equations

$$\frac{d(R + hr)}{dy_1} = 0, \quad \frac{d(R + hr)}{dy_2} = 0 . ., \quad \frac{d(R + hr)}{dy_m} = 0 . . . . (76),$$

be regarded as merging into an equal number of independent equations,

$$\frac{y_1}{n_1} = \frac{y_2}{n_2} . . = \frac{y_m}{n_m}, \quad \theta(R + hr) = 0 \quad . . . . (77),$$

$n_1\, n_2 . . n_m$ being functions of the coefficients in $R + hr$. Now the two systems of equations (74) and (76) are mutually dependent, hence are also the systems (75) and (77). The conse-

quences which follow from the mutual dependence of the two last equations of these two systems have already been examined. The remaining ones, in their mutual dependence, are scarcely of less importance, as enabling us to determine the linear theorems connecting $x_1, x_2 \dots x_m$ with $y_1, y_2 \dots y_m$.

12. We see, in fact, that when the values of $x_1, x_2 \dots x_m$, are chosen proportionals to $l_1, l_2 \dots l_m$, respectively, the simultaneous values of $y_1, y_2 \dots y_m$, will be proportional to $n_1, n_2 \dots n_m$. To determine the actual magnitudes of the values of $y_1, y_2 \dots y_m$, corresponding to an assumed series of values of $x_1, x_2 \dots x_m$, another equation is evidently necessary, and for this purpose either of the primitive equations, $q = r$, or $Q = R$, is sufficient. We choose the former, and suppose that the substitution in q of $l_1, l_2 \dots l_m$, in the place of $x_1, x_2 \dots x_m$, gives a result L; and that the substitution of $n_1, n_2 \dots n_m$, for $y_1, y_2 \dots y_m$, in r, gives N; then it is manifest, that since q and r are homogeneous and of the n^{th} degree, the equation $q = r$ will be satisfied, as well as the $m - 1$ first equations of (75) and of (77), by the assumptions

$$x_1 = \frac{l_1}{\sqrt[n]{L}}, \quad x_2 = \frac{l_2}{\sqrt[n]{L}} \dots x_m = \frac{l_m}{\sqrt[n]{L}} \dots\dots (78),$$

$$y_1 = \frac{n_1}{\sqrt[n]{N}}, \quad y_2 = \frac{n_2}{\sqrt[n]{N}} \dots y_m = \frac{n_m}{\sqrt[n]{N}} \dots\dots (79),$$

which are therefore a set of simultaneous values of the $2m$ variables in question.

Now $l_1, l_2 \dots l_m, n_1, n_2 \dots n_m$, involve h; as many different values as are therefore assigned to that quantity, so many sets of simultaneous values will the above equations (78) and (79) afford for the variables $x_1, x_2 \dots x_m, y_1, y_2, \dots y_m$; the successive substitution of which in the linear forms

$$\left.\begin{aligned} x_1 &= \lambda_1 y_1 + \lambda_2 y_2 \dots + \lambda_m y_m \\ x_2 &= \mu_1 y_1 + \mu_2 y_2 \dots + \mu_m y_m \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_m &= \rho_1 y_1 + \rho_2 y_2 \dots + \rho_m y_m \end{aligned}\right\} \dots\dots (80),$$

will give equations serving to determine by linear elimination, the values of the constants

$$\lambda_1, \mu_1, \rho_1 \dots \lambda_m, \mu_m \dots \rho_m.$$

The application of this method to (69) and (70), leads to the results

$$x = x' + y',$$
$$y = x',$$

which are easily verified. The values of h, for this case, as determined by the solution of (71), are -1, $+\frac{1}{3}$, $-\frac{2}{3}$; and the formulæ to be employed in conjunction with either of the primitive equations (69) and (70), are

$$\frac{x}{A_1 D_1 - B_1 C_1} = \frac{y}{2(B_1^2 - A_1 C_1)} \cdots\cdots (81),$$

$$\frac{x'}{A_1' D_1' - B_1' C_1'} = \frac{y'}{2(B_1'^2 - A_1' C_1')} \cdots\cdots (82),$$

as will be seen on reference to (19).

It might be presumed, that as the values of h are in some cases more numerous, and in others, from equality of the roots, less so, than would appear to be necessary for the formation of the different sets of simultaneous values of the variables to be employed in the above process, we should in the former case arrive at superfluous results, and in the latter be compelled to have recourse to a different method of solution, for the discovery of the linear relations. How far these anticipations might prove correct I am not prepared to say, but I apprehend that under either of these circumstances, as well as under the supposed condition of h receiving imaginary values, an answering peculiarity will be found in the relations sought, rendering the solution possible and definite.

13. The functions $\theta(Q)$ and $\theta(R)$ may be shewn to possess many remarkable properties, both individually and in mutual relation. Of these the one I am about to demonstrate is perhaps the most important. It has been established in this paper, that when any two homogeneous functions, Q and q, with the same variables, and of the same degree, are by a common system of linear relations transformed into R and r, then

$$\frac{\theta(Q)}{\theta(q)} = \frac{\theta(R)}{\theta(r)} \cdots\cdots\cdots (83).$$

Let the ratio of $\theta(R)$ to $\theta(Q)$ be represented by E, so that $\theta(Q) = \frac{\theta(R)}{E}$; then by (83) also $\theta(q) = \frac{\theta(r)}{E}$. The nature of the function E it will be necessary to examine.

It is in the first place evident that E cannot in any way functionally depend on the constants in Q and R, or in q and r, otherwise the equations

$$\theta(Q) = \frac{\theta(R)}{E}, \quad \theta(q) = \frac{\theta(r)}{E},$$

would suppose a relation among the constants in Q and q, which are entirely independent. Hence E can only involve the constants contained in the linear theorems. We proceed to determine its form in one or two simple cases.

Let Q_2 be a homogeneous function of the second degree, with two variables, x and y; and let the linear relations by which Q_2 is supposed to be transformed into R_2, be

$$\left.\begin{aligned} x &= mx' + ny' \\ y &= m'x' + n'y' \end{aligned}\right\} \qquad (84).$$

Since E is independent of the constants in Q_2, we may assume the values of those constants at pleasure, provided that our assumptions do not cause $\theta(Q_2)$ to vanish. Let then $Q_2 = x^2 + y^2$, whence $\theta(Q_2) = 1$; also by substitution of (84),

$$R_2 = (m^2 + m'^2)\,x'^2 + 2\,(mn + m'n')\,x'y' + (n^2 + n'^2)\,y'^2,$$

$$\therefore\ \theta(R_2) = (m^2 + m'^2)(n^2 + n'^2) - (mn + m'n')^2 = (mn' - m'n)^2,$$

$$E = \frac{\theta(R_2)}{\theta(Q_2)} = (mn' - m'n)^2.$$

Hence if Q_2 be any homogeneous function of the second degree, transformed by (84) to R_2, we shall always have

$$\theta(Q_2) = \frac{\theta(R_2)}{(mn' - m'n)^2} \qquad (85).$$

If Q_3 be a homogeneous function of the third degree, similarly transformed by (84) into R_3, we shall, after an analogous but very complicated process, find

$$\theta(Q_3) = \frac{\theta(R_3)}{(mn' - m'n)^6} \qquad (86).$$

The singularity of these results has led me to investigate the general law on which they depend, and I have arrived at the following theorem.

B. *If* Q$_n$ *be a homogeneous function of the* n$^{\text{th}}$ *degree, with* m *variables, which by the linear theorems* (80) *is transformed into* R$_n$, *a similar homogeneous function; and if* γ *represent the degree of* $\theta(\mathrm{Q}_n)$ *and* $\theta(\mathrm{R}_n)$, *then*

$$\theta(Q_n) = \frac{\theta(R_n)}{E^{\frac{\gamma n}{m}}} \qquad (87),$$

E being the result obtained by the elimination of the variables from the second members of the linear theorems (80), equated to 0.

From (85) and (86) we derive the important theorem,

$$\frac{\theta(Q_3)}{\{\theta(Q_2)\}^3} = \frac{\theta(R_3)}{\{\theta(R_2)\}^3} \ldots\ldots (88),$$

a theorem which, by (87), is easily extended to general indices. The application of this result we shall have occasion to exemplify in the researches with which the second part of this memoir will be occupied.

Resuming our former notation, we may observe from (87), that if $E = 0$, then either $\theta(Q)$ and $\theta(q)$ become infinite, or $\theta(R)$ and $\theta(r)$ vanish. Under either of these circumstances, the equations of the general solution (32), (33), (34), disappear, or are greatly modified. By the nature of linear elimination it is evident, that the condition $E = 0$ and the condition $F(\lambda_1, \mu_1, \rho_1 \ldots \lambda_m, \mu_m, \rho_m) = 0$ of (9), are identical, a circumstance which verifies the remark in 83, relative to the latter condition.

Minster Yard, Lincoln, April 28th, 1841.

II.—ON THE SURD ROOTS OF EQUATIONS.

By R. Moon, M.A. Fellow of Queens' College.

Lemma. If we have the equation

$$A + Bx^{\frac{1}{m}} + Cx^{\frac{2}{m}} + \ldots + Lx^{\frac{m-1}{m}} = 0,$$

where $A, B, C \ldots L$ and x are all rational quantities, and $x^{\frac{1}{m}}, x^{\frac{2}{m}} \ldots x^{\frac{m-1}{m}}$, are all surds, we must have

$$A = 0, \quad B = 0, \quad C = 0, \ldots L = 0.$$

Suppose the rule to apply to the equation

$$a + a_1 x^{\frac{1}{m}} + a_2 x^{\frac{2}{m}} + \ldots + a_{m-2} x^{\frac{m-2}{m}} = 0 \ \ldots\ldots (1),$$

it must then also apply to the equation

$$a + a_1 x^{\frac{1}{m}} + a_2 x^{\frac{2}{m}} + \ldots + a_{m-2} x^{\frac{m-2}{m}} + a_{m-1} x^{\frac{m-1}{m}} = 0.$$

For from this last we have

$$aa_{m-2} + a_1 a_{m-2} x^{\frac{1}{m}} + a_2 a_{m-2} x^{\frac{2}{m}} + \ldots + a^2_{m-2} x^{\frac{m-2}{m}} + a_{m-2} a_{m-1} x^{\frac{m-1}{m}} = 0,$$

and

$$aa_{m-1}x^{\frac{1}{m}} + a_1 a_{m-1}x^{\frac{2}{m}} + a_2 a_{m-1}x^{\frac{3}{m}} + \ldots + a_{m-2}a_{m-1}x^{\frac{m-1}{m}} + a^2_{m-1}x = 0,$$

or subtracting,

$$\left.\begin{array}{r} aa_{m-2} - a^2_{m-1}x + (a_1 a_{m-2} - aa_{m-1})\,x^{\frac{1}{m}} + (a_2 a_{m-2} - a_1 a_{m-1})\,x^{\frac{2}{m}} \\ + \,\&\text{c.} \\ + \,(a^2_{m-2} - a_{m-3}a_{m-1})\,x^{\frac{m-2}{m}} \end{array}\right\} = 0 \;.\,. (2);$$

whence by our hypothesis we have

$$\begin{aligned} aa_{m-2} - a^2_{m-1}x &= 0, \\ a_1 a_{m-2} - aa_{m-1} &= 0, \\ a_2 a_{m-2} - a_1 a_{m-1} &= 0, \\ \ldots\ldots\ldots\ldots &= 0, \\ a^2_{m-2} - a_{m-3}a_{m-1} &= 0; \end{aligned}$$

from which series of equations we obtain, by eliminating $a, a_1, a_2 \ldots . a_{m-3}$,

$$x = \left(\frac{a_{m-2}}{a_{m-1}}\right)^m,$$

an equation which comprises all the roots of equation (2).

Now it is evident that if we eliminate $x^{\frac{m-1}{m}}$ from the equation

$$a - e\left(\frac{a_{m-2}}{a_{m-1}}\right)^m + ex + a_1 x^{\frac{1}{m}} + a_2 x^{\frac{2}{m}} + \ldots + a_{m-1}x^{\frac{m-1}{m}} = 0,$$

(which we will represent by $u = 0$,) by the same process which we adopted with respect to (1), that we shall in like manner arrive at the result

$$x = \left(\frac{a_{m-2}}{a_{m-1}}\right)^m \ldots\ldots\ldots (3);$$

and this last equation will include all the roots of

$$u\,(a_{m-2} - a_{m-1}x^{\frac{1}{m}}) = 0,$$

and therefore of $u = 0$.

If we represent the roots of the equation $x^m - 1 = 0$, by $1, \mathfrak{a}_1, \mathfrak{a}_2, \ldots . \mathfrak{a}_{m-1}$, it is clear that u will be divisible by

$$A = \left(x^{\frac{1}{m}} - \mathfrak{a}_1\frac{a_{m-2}}{a_{m-1}}\right)\left(x^{\frac{1}{m}} - \mathfrak{a}_2\frac{a_{m-2}}{a_{m-1}}\right) \ldots \left(x^{\frac{1}{m}} - \mathfrak{a}_{m-1}\frac{a_{m-2}}{a_{m-1}}\right),$$

without a remainder. Now

$$u = e \cdot \left(x - \frac{a_{m-2}}{a_{m-1}}\right)^m + \phi\left(x^{\frac{1}{m}}\right),$$

where ϕ represents a function of $m-1$ dimensions; and if we divide $e \cdot \left(x - \frac{a_{m-2}}{a_{m-1}}\right)^m$ by A, the result is $\left(x^{\frac{1}{m}} \mp \alpha\right) e$; whence is plain that $\phi\left(x^{\frac{1}{m}}\right) = A \times \text{const.} = Aa_{m-1}$; hence the roots of the equation $u = 0$ are

$$\left(e \cdot \frac{a_{m-2}}{a_{m-1}} - a_{m-1}\right)\left(\alpha_1 \frac{a_{m-2}}{a_{m-1}}\right)\left(\alpha_2 \frac{a_{m-2}}{a_{m-1}}\right) \cdots \left(\alpha_{m-1} \frac{a_{m-2}}{a_{m-1}}\right).$$

But we have before seen that the roots of $u = 0$ are included among the roots of (3), therefore $\left(e \cdot \frac{a_{m-2}}{a_{m-1}} - a_{m-1}\right)$ must be identical with some one of the quantities

$$\frac{a_{m-2}}{a_{m-1}}, \quad \alpha_1 \frac{a_{m-2}}{a_{m-1}}, \quad \alpha_2 \frac{a_{m-2}}{a_{m-1}}, \ldots \alpha_{m-1} \frac{a_{m-2}}{a_{m-1}},$$

which is impossible, since e may be any rational quantity whatever.

Suppose the rule to apply to the equation

$$a + a_1 x^{\frac{1}{m}} + a_2 x^{\frac{2}{m}} + \ldots + a_{m-3} x^{\frac{m-3}{m}} = 0,$$

it will also apply to the equation

$$a + a_1 x^{\frac{1}{m}} + a_2 x^{\frac{2}{m}} + \ldots + a_{m-3} x^{\frac{m-3}{m}} + a_{m-2} x^{\frac{m-2}{m}} = 0 \ldots\ldots (4).$$

For from this last we have

$$aa_{m-3} x^{\frac{1}{m}} + a_1 a_{m-3} x^{\frac{2}{m}} + \ldots + a^2_{m-3} x^{\frac{m-2}{m}} + a_{m-2} a_{m-3} x^{\frac{m-1}{m}} = 0,$$

and $$aa_{m-2} x^{\frac{2}{m}} + a_1 a_{m-2} x^{\frac{3}{m}} + \ldots + a_{m-2} a_{m-3} x^{\frac{m-1}{m}} + a^2_{m-2} x = 0,$$

and subtracting we have

$$\left.\begin{array}{l} -a^2_{m-2} x + aa_{m-3} x^{\frac{1}{m}} + (a_1 a_{m-3} - aa_{m-2})\, x^{\frac{2}{m}} \\ \qquad + \ldots \\ \qquad + (a^2_{m-3} - a_{m-2} a_{m-4})\, x^{\frac{m-2}{m}} \end{array}\right\} = 0;$$

and eliminating $x^{\frac{m-2}{m}}$ from this last by means of (4), (which we will represent by $u = 0$,) the resulting equation will be

$$u\left(a_{m-3} - a_{m-2} x^{\frac{1}{3}}\right) x^{\frac{1}{3}} \times \text{const.} = 0 \ldots (5);$$

whence, by means of our hypothesis, it is plain we can find

$$x = \phi\left(a_{m-2} a_{m-3}\right) \ldots\ldots\ldots\ldots\ldots (6),$$

an equation whose roots comprise all the finite roots of (5), and which gives m different values of $x^{\frac{1}{m}}$; whence it follows that the equation $u = 0$ must have $m - 1$ different roots, which is absurd.

It may be proved in a manner precisely similar to that adopted in the latter of the two above cases, that if the rule applies to the equation

$$a + a_1 x^{\frac{1}{m}} + a_2 x^{\frac{2}{m}} + \ldots + a_{m-n}{}^{\frac{m-n}{m}} = 0,$$

it will also apply to the equation

$$a + a_1 x^{\frac{1}{m}} + a_2 x^{\frac{2}{m}} + \ldots + a_{m-n} x^{\frac{m-n}{m}} + a_{m-n+1} x^{\frac{m-n+1}{m}} = 0.$$

But the rule holds for the equation

$$a + a_1 x^{\frac{1}{m}} = 0;$$

hence it holds generally.

We are now enabled to shew that, if $a + b^{\frac{1}{m}}$ be a root of an equation, where a and b are rational, but $b^{\frac{1}{m}}$ is a surd; $a + \alpha b^{\frac{1}{m}}$ is also a root where α is any root of the equation $x^m - 1 = 0$.

Take the case of

$$x^5 + px^4 + qx^3 + rx^2 + sx + t = 0.$$

and let the result of the substitution of $a + b^{\frac{1}{3}}$ in this equation be

$$b^{\frac{5}{3}} + A_1 b^{\frac{4}{3}} + A_2 b + A_3 b^{\frac{2}{3}} + A_4 b^{\frac{1}{3}} + A_5 = 0;$$

whence, by our Lemma,

$$b^{\frac{5}{3}} + A_3 b^{\frac{2}{3}} = 0 \ldots\ldots (1),$$

$$A_1 b^{\frac{4}{3}} + A_4 b^{\frac{1}{3}} = 0 \ldots\ldots (2),$$

$$A_2 b + A_5 = 0 \ldots\ldots (3).$$

Let α be a root of the equation $x^3 - 1 = 0$; then we have

$$(1) \times \alpha^2 + (2) \times \alpha + (3) = 0,$$

or (observing that $\alpha^3 = 1$,) we have, making the necessary arrangement,

$$b^{\frac{5}{3}}\alpha^5 + A_1 b^{\frac{4}{3}}\alpha^4 + A_2 b\alpha^3 + A_3 b^{\frac{2}{3}}\alpha^2 + A_4 b^{\frac{1}{3}}\alpha + A_5 = 0;$$

that is, $a + \alpha b^{\frac{1}{m}}$ is a root; and the same proof evidently applies generally. Hence, if an equation has a root $a + b^{\frac{1}{m}}$, it has a corresponding factor

$$(x - a)^m - b = 0.$$

It may be observed, that the occurrence of these groups of roots will, in certain cases, facilitate the solution: thus in the present case the solution might be effected by means of an equation of four dimensions.

It will be found also, that if m be the denominator of the surd, and n the number of dimensions of the equation, and

$$m > \frac{n+1}{2}, \text{ where } n \text{ is odd,}$$

$$\text{or } m > \frac{n}{2}, \text{ where } n \text{ is even;}$$

the group of roots $a + b^{\frac{1}{m}}$, will introduce no more complexity than a single root; that is, the equation may be solved by means of one of $n - m + 1$ dimensions.

It only remains to add, that if

$$a + b^{\frac{1}{2}} + c^{\frac{1}{3}}$$

be a root of an equation, there will be six roots depending on the same irrational parts, which are comprised under the form

$$a \pm b^{\frac{1}{2}} + \alpha c^{\frac{1}{3}},$$

where α is a root of the equation $x^3 - 1 = 0$; for whatever be the number of roots depending upon $b^{\frac{1}{2}}$, $c^{\frac{1}{3}}$, it is clear that if when $b = 0$, a root $a + c^{\frac{1}{3}}$ occur, we must have corresponding to it two others, $a + \alpha c^{\frac{1}{3}}$, $a + \alpha^2 c^{\frac{1}{3}}$; hence, if a root occur in the proposed equation, of the form $a + c^{\frac{1}{3}} + \phi$, where ϕ vanishes when $b = 0$, there must be corresponding to it the roots $a + \alpha c^{\frac{1}{3}} + \phi$, $a + \alpha^2 c^{\frac{1}{3}} + \phi$. In like manner it may be shewn that whenever a root $a + b^{\frac{1}{2}} + \phi$ occurs in the pro-

posed equation where ϕ vanishes when $c^{\frac{1}{3}} = 0$ there must be corresponding to it the root $a - b^{\frac{1}{2}} + \phi$. Hence, corresponding to

$$a + b^{\frac{1}{2}} + c^{\frac{1}{3}}, \text{ we have } a + b^{\frac{1}{2}} + \mathbf{a}c^{\frac{1}{3}},$$

$$a + b^{\frac{1}{2}} + \mathbf{a}^2 c^{\frac{1}{3}};$$

corresponding to

$$a + b^{\frac{1}{2}} + \mathbf{a}c^{\frac{1}{3}}, \text{ we have } a - b^{\frac{1}{2}} + \mathbf{a}c^{\frac{1}{3}},$$

$$a + b^{\frac{1}{2}} + \mathbf{a}^2 c^{\frac{1}{3}}, \; \ldots\ldots\ldots \; a - b^{\frac{1}{2}} + \mathbf{a}^2 c^{\frac{1}{3}}.$$

And generally, if we have an equation having a root

$$a + b^{\frac{1}{m}} + c^{\frac{1}{n}} + d^{\frac{1}{p}} + \ldots.$$

it must have for a root

$$a + \beta b^{\frac{1}{m}} + \gamma c^{\frac{1}{n}} + \delta d^{\frac{1}{p}} + \ldots.$$

where β is any root of the equation $x^m - 1 = 0$,

γ .... $x^n - 1 = 0$,

δ .... $x^p - 1 = 0$,

.... $= 0$,

the number of roots in the last case will be $= m \,.\, n \,.\, p \ldots.$ and it may easily be shewn that the factor to which they will jointly give rise will be rational.

III.—NOTE ON A PASSAGE IN FOURIER'S HEAT.*

In finding the motion of heat in a sphere, Fourier expands a function Fx, arbitrary between the limits $x = 0$ and $x = X$, in a series of the form

$$a_1 \sin n_1 x + a_2 \sin n_2 x + \text{\&c.}$$

where n_1, n_2, &c. are the successive roots of the equation

$$\frac{\tan nX}{nX} = 1 - hX.$$

Now Fourier gives no demonstration of the possibility of this

* From a Correspondent.

expansion, but he merely determines what the coefficients a_1, a_2, &c. would be, if the function were represented by a series of this form. Poisson arrives, by another method, at the same conclusion as Fourier, and then states this objection to Fourier's solution; but, as is remarked by Mr. Kelland, (*Theory of Heat*, p. 81, Note,) he "does not appear, as far as I can see, to get over the difficulty." The writer of the following article hopes that the demonstration in it will be considered as satisfactory, and consequently as removing the difficulty.

$$\text{Let} \quad n_i X = \varepsilon_i, \quad \frac{\pi x}{X} = x', \quad \text{and} \quad Fx = fx'.$$

Then the preceding series will take the form

$$a_1 \sin \frac{\varepsilon_1 x}{\pi} + a_2 \sin \frac{\varepsilon_2 x}{\pi} + \&c.,$$

the accents being omitted above x.

Now it is shewn by Fourier, that

$$\varepsilon_i = \left(\frac{2i-1}{2} - c_i\right)\pi,$$

where c_i is always less than $\frac{1}{2}$, and is equal to 0, when i is infinitely great. Hence the series becomes

$$a_1 \sin\left(\frac{1}{2} - c_1\right)x + a_2 \sin\left(\frac{3}{2} - c_2\right)x + \&c. \ldots . (a).$$

Now it is easily shown, from the fact that any function of x can be represented, between the limits 0 and π, by a series of either sines or cosines of multiples of x, that it may be represented, between the same limits, by a series of the form

$$A \sin \frac{1}{2} x + B \sin \frac{3}{2} x + \&c.$$

Hence each of the quantities

$$\sin\left(\frac{1}{2} - c_1\right)x, \quad \sin\left(\frac{3}{2} - c_2\right)x, \ \&c.,$$

can be developed in a series of this form. We may consequently assume $\sin\left(\frac{1}{2} - c_1\right)x$ equal i terms of a series of sines of odd multiples of $\frac{1}{2}x$, together with a quantity, ${}^i e_1$; $\sin\left(\frac{3}{2} - c_2\right)x$ equal to i terms of a similar series, together with a quantity ${}^i e_2$; and so with all the terms of the series (a), up to the term $\sin\left(\frac{2i-1}{2} - c_i\right)x$, which may be assumed

equal to i terms, together with a quantity ${}^{i}e_i$; and it is readily seen, that each of the quantities ${}^{i}e_1$, ${}^{i}e_2$, ... ${}^{i}e_i$, is infinitely small when i is infinitely great. Hence we shall have

$$a_1 \sin\left(\frac{1}{2} - c_1\right)x + a_2 \sin\left(\frac{3}{2} - c_2\right)x + \ldots + a_i \sin\left(\frac{2i-1}{2} - c_i\right)x$$

$$= A_1 \sin\frac{1}{2}x + A_2 \sin\frac{3}{2}x + \ldots + A_i \sin\frac{2i-1}{2}x$$

$$+ a_1\,{}^{i}e_1 + a_2\,{}^{i}e_2 + \ldots + a_i\,{}^{i}e_i,$$

A_1, A_2, ... A_i, being known, in terms of a_1, a_2, ... a_i. Hence, conversely, any series,

$$A_1 \sin\frac{1}{2}x + A_2 \sin\frac{3}{2}x + \ldots + A_i \sin\frac{2i-1}{2}x,$$

where A_1, A_2, ... A_i, are arbitrary, may be represented by another series of the form

$$a_1\left\{\sin\left(\frac{1}{2} - c_1\right)x - {}^{i}e_1\right\} + a_2\left\{\sin\left(\frac{3}{2} - c_2\right)x - {}^{i}e_2\right\} + \ldots$$

$$+ a_i\left\{\sin\left(\frac{2i-1}{2}x - c_i\right) - {}^{i}e_i\right\},$$

where $a_1, a_2, \ldots a_i$ are determined, in terms of $A_1, A_2, \ldots A_i$, by i equations, giving the latter quantities in terms of the former.

Let now $i = \infty$; then each of the quantities ${}^{i}e_1$, ${}^{i}e_2$, ... ${}^{i}e_i$, will vanish, and it will follow that any series,

$$A_1 \sin\frac{1}{2}x + A_2 \sin\frac{3}{2}x + \&c.,$$

may be represented by a series of the form

$$a_1 \sin\left(\frac{1}{2} - c_1\right)x + a_2 \sin\left(\frac{3}{2} - c_2\right)x + \&c.$$

Now any function, fx, can be represented, between the limits $x = 0$ and $x = \pi$, by the former series, and consequently by the latter also, between the same limits. But the latter series is equal to

$$a_1 \sin\frac{\varepsilon_1 x}{\pi} + a_2 \sin\frac{\varepsilon_2 x}{\pi} + \&c.;$$

and hence fx can be represented, between the limits 0 and π, by this series; and therefore it follows, that any function, Fx, can be represented, between the limits 0 and X, by the series

$$a_1 \sin n_1 x + a_2 \sin n_2 x + \&c.$$

P. Q. R.

IV.—NOTE ON A SOLUTION OF A CUBIC EQUATION.

By J. COCKLE, B.A.

To the Editor of the Cambridge Mathematical Journal.

SIR,—Permit me to add a few remarks to the Solution of a Cubic Equation, which you did me the honour of inserting in your twelfth number.

Let α', β', γ', be the three roots of the original equation $f(x) = 0$; then the roots of the transformed equation $f(y) = 0$, will be $\alpha' - z$, $\beta' - z$, $\gamma' - z$, which let equal α, β, γ, respectively. By the known law of formation of the coefficients of an equation, the relation, among the coefficients of $f(y) = 0$, $B^2 = 3.AC$ is the same as

$$(\alpha\beta + \alpha\gamma + \beta\gamma)^2 = 3 \,.\, \alpha\beta\gamma \,.\, (\alpha + \beta + \gamma);$$

which, by expanding, transposing, and dividing by $\alpha^2, \beta^2, \gamma^2$, becomes

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} \;\ldots\ldots\; (1).$$

Solving this as a quadratic in $\frac{1}{\alpha}$, we find that part of the value of $\frac{1}{\alpha}$, which is under the radical sign, to be

$$-\frac{3}{4} \,.\, \left(\frac{1}{\beta} \sim \frac{1}{\gamma}\right)^2,$$

which, being essentially negative if β and γ are real (unless $\beta = \gamma$), shews that the equation (1) is inconsistent with the reality of *all* the quantities α, β, γ; if therefore α', β', γ', be all possible, z will be impossible, and the expression for x will assume an impossible form, unless two of its roots be equal; but when two of the roots of $f(x) = 0$ are impossible, (1) may be satisfied by a possible value of z, and the possible root at once exhibited, as in the examples.

On solving the equation in z, that part of its value which is under the radical sign is

$$\frac{(ab - qc)^2 - 4\,(a^2 - 3b)\,(b^2 - 3ac)}{4\,(a^2 - 3b)^2};$$

and, the denominator being a complete square, the condition for z being possible is that the numerator should be not less than zero, or (reducing) that

$$81c^2 + 12b^3 + 12a^3c - 3a^2b^2 - 54abc \;\overline{\overline{>}}\; 0,$$

or, dividing by $3 \times 4 \times 27$,

$$\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3 + c\left(\frac{a}{3}\right)^3 - \frac{ab}{4 \times 27}(ab + 18c) \mathrel{\substack{=\\>}} 0,$$

which is the condition for z being possible, that is, of there being one and only one real root to the equation; or of all the roots being real and two equal; which last will be the case if the left-hand side of the above equals 0. If the above does not hold, all the roots are real and unequal.

Although there are two values of z, and three of the radical, which enters into the expression for x, yet these give only three values for x; since, obviously, the two values of z admit of no combinations with one another, but whichever value we take is used as if the other had no existence. It is indifferent which value is selected; thus, in the second example given of the method, the value $z = \frac{10}{4}$ gives the same result as $z = 1$.

Trin. Coll. Camb., May 19, 1841.

V.—ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS.

By B. Bronwin.

Let $(1 - x^2)\dfrac{d^2y}{dx^2} + my = 0$, $m = p(p-1)$, m and p integers

If $y = \Sigma a_n x^n$, we have for the determination of a_n the equation

$$n(n-1)a_n = \{(n-2)(n-3) - m\}\, a_{n-2}.$$

I shall call this the scale of the equation for the sake of convenience.

Making $n = 0, 2, 4$, &c. we have $a_{-2} = 0$, $a_{-4} = 0$, &c. and

$$a_2 = -\frac{m}{2}a_0,\ a_4 = \frac{m(m-1.2)}{2.3.4}a_0,\ a_6 = -\frac{m(m-1.2)(m-3.4)}{2.3.4.5.6}a_0, \text{ \&c.}$$

and making $n = 1, 3$, &c., $a_{-1} = 0$, $a_{-3} = 0$, &c.

$$a_3 = -\frac{m}{2.3}a_1,\ a_5 = \frac{m(m-2.3)}{2.3.4.5}a_1,\ a_7 = -\frac{m(m-2.3)(m-4.5)}{2.3.4.5.6.7}a_1, \text{ \&c.}$$

Consequently

$$y = a_0\left\{1 - \frac{m}{2}x^2 + \frac{m(m-1.2)}{2.3.4}x^4 - \text{\&c.}\right\} + a_1\left\{x - \frac{m}{2.3}x^3 + \frac{m(m-2.3)}{2.3.4.5}x^5 - \text{\&c.}\right\}$$

This is the complete integral of the proposed. One of the series which it contains will always terminate in the case supposed, and give a particular integral in finite terms, by means of which the other particular integral may be found in finite terms also.

Let $m=p=2$; we have $y=a_0(1-x^2)=C(1-x^2)$, a particular integral. Make $y=(1-x^2)z$. Putting this value in the proposed, we find

$$z=C\int\frac{dx}{(1-x^2)^2}=\tfrac{1}{4}C\left(\frac{2x}{1-x^2}+\log\frac{1+x}{1-x}\right)+C',$$

$$\text{and } y=C''\left\{2x+(1-x^2)\log\frac{1+x}{1-x}\right\}+C'(1-x^2),$$

for the complete integral.

Again, let $p=3$, or $m=6$; then $y=Cx(1-x^2)$, a particular integral. If $y=x(1-x^2)z$, we find

$$z=C\int\frac{dx}{x^2(1-x^2)^2}=\tfrac{3}{4}C\left\{\frac{2x^2-\frac{4}{3}}{x(1-x^2)}+\log\frac{1+x}{1-x}\right\}+C',$$

$$\text{and } y=C''\left\{2x^2-\tfrac{4}{3}+x(1-x^2)\log\frac{1+x}{1-x}\right\}+C'x(1-x^2),$$

for the complete integral.

In the general case the second particular integral may be found in finite terms, thus:

$$\text{Make } y=u+v\log\frac{1+x}{1-x},$$

where v is the particular integral which terminates. This value substituted in the proposed, gives

$$(1-x^2)\frac{d^2u}{dx^2}+mu+4\frac{dv}{dx}+\frac{4x}{1-x^2}v=0;$$

or if $v=(1-x^2)w$,

$$(1-x^2)\frac{d^2u}{dx^2}+mu+4(1-x^2)\frac{dw}{dx}-4xw=0.$$

If $u=\Sigma a_nx^n$, $w=\Sigma b_nx^n$, the scale of this is

$$n(n-1)a_n+\{m-(n-2)(n-3)\}a_{n-2}+4(n-1)b_{n-1}-4(n-2)b_{n-3}=0.$$

We now want the value of w. To obtain it we make $y=(1-x^2)z$. This value substituted in the proposed gives

$$(1-x^2)\frac{d^2z}{dx^2}-4x\frac{dz}{dx}+(m-2)z=0,$$

of which the scale is

$$n(n-1)\,b_n = \{n(n-1)-m\}\,b_{n-2},$$

which gives

$$z = b_0\left\{1 - \frac{m-1.2}{2}\,x^2 + \frac{(m-1.2)(m-3.4)}{2.3.4}\,x^4 - \&c.\right\}$$

$$+\, b_1\left\{x - \frac{m-2.3}{2.3}\,x^3 + \frac{(m-2.3)(m-4.5)}{2.3.4.5}\,x^5 - \&c.\right\}.$$

The particular value of this which terminates gives w.

Returning now to the former equation, all the values of b_{-1}, b_{-2}, &c. and of a_{-1}, a_{-2}, &c. are nothing. And since we only want one arbitrary, which will be b_0 or b_1, we shall have a_0 or a_1 to assume at pleasure, but we shall not know how to assume it so as to make the series terminate. We shall therefore begin at the further extremity of the series, where $b_p = 0$, $b_{p+2} = 0$, &c.

If we make $a_{p+1} = 0$, we shall have a_{p+3}, a_{p+5}, &c. $= 0$, and we shall determine a_{p-1}, a_{p-3}, &c. in a retrograde order, and we shall have the value of u expressed by a series which terminates.

But we have z, or rather a particular value of z, by a descending series. Thus,

$$z = a_{p-2}\left\{x^{p-2} - \frac{(p-2)(p-3)}{2(2p-3)}\,x^{p-4} + \&c.\right\}.$$

Taking this for w, and making $a_{p+1} = 0$, we shall easily find u by a descending series which terminates.

If we integrate the proposed term by term successively, we have

$$(1-x^2)\frac{d^2y_1}{dx^2} + 2x\frac{dy_1}{dx} + (m-2)\,y_1 = 0, \quad y_1 = \int y\,dx;$$

$$(1-x^2)\frac{d^2y_2}{dx^2} + 4x\frac{dy_2}{dx} + (m-2.3)\,y_2 = 0, \quad y_2 = \int y_1\,dx; \ \&c.$$

$$(1-x^2)\frac{d^2y_{p-1}}{dx^2} + 2(p-1)\,x\,\frac{dy_{p-1}}{dx} = 0,$$

which is integrable, and will give a particular integral of the proposed.

If now we differentiate successively, we find

$$(1-x^2)\frac{d^2y_1}{dx^2} - 2x\frac{dy_1}{dx} + my_1 = 0, \quad y_1 = \frac{dy}{dx};$$

$$(1-x^2)\frac{d^2y_2}{dx^2} - 4x\frac{dy_2}{dx} + (m-2)\,y_2 = 0, \quad y_2 = \frac{dy_1}{dx};$$

$$(1-x^2)\frac{d^2y_3}{dx^2} - 6x\frac{dy_3}{dx} + (m-2.3)\,y_3 = 0, \quad y_3 = \frac{dy_2}{dx}; \text{ \&c.}$$

$$(1-x^2)\frac{d^2y_p}{dx^2} - 2px\frac{dy_p}{dx} = 0.$$

This will give the other particular integral of the proposed: and it is obvious that the integrals of all the preceding will easily be derived from the integral of the proposed.

We now proceed to a second example.

Let $(1+x^2)\dfrac{d^2y}{dx^2} - my = 0$, m as before. The scale of this is

$$n(n-1)\,a_n = \{m-(n-2)(n-3)\}\,a_{n-2},$$

which gives

$$y = a_0\left\{1 + \frac{m}{2}x^2 + \frac{m(m-1.2)}{2.3.4}x^4 + \text{\&c.}\right\} + a_1\left\{x + \frac{m}{2.3}x^3 + \frac{m(m-2.3)}{2.3.4.5}x^5 + \text{\&c.}\right\}$$

One of these series will terminate and give a particular integral in finite terms. Let $p = m = 2$, then $y = C(1+x^2)$, a particular integral. Make $y = (1+x^2)z$, and we find by substitution in the proposed,

$$z = C\int\frac{dx}{(1+x^2)^2} = \tfrac{1}{2}C\left\{\frac{x}{1+x^2} + \tan^{-1}x\right\} + C',$$

$$\text{and } y = C''\{x + (1+x^2)\tan^{-1}x\} + C'(1+x^2),$$

for the complete integral.

Again, let $p = 3$, or $m = 6$; then $y = Cx(1+x^2)$, a particular integral. And if we make $y = x(1+x^2)z$, we find

$$z = C\int\frac{dx}{x^2(1+x^2)^2} = -\frac{3}{2}C\left\{\frac{x^2+\frac{2}{3}}{x(1+x^2)} + \tan^{-1}x\right\} + C',$$

$$\text{and } y = C''\{x^2 + \tfrac{2}{3} + x(1+x^2)\tan^{-1}x\} + C'x(1+x^2),$$

for the complete integral.

By proceeding as in the last example, putting $\tan^{-1}x$ instead of $\log\dfrac{1+x}{1-x}$, we shall find the second particular integral in the general case in finite terms. Also,

$$(1+x^2)\frac{d^2y_1}{dx^2} - 2x\frac{dy_1}{dx} - (m-2)\,y_1 = 0, \quad y_1 = \int y\,dx;$$

$$(1+x^2)\frac{d^2y_2}{dx^2} - 4x\frac{dy_2}{dx} - (m-2.3)\,y_2 = 0, \quad y_2 = \int y_1\,dx; \text{ \&c.}$$

$$(1+x^2)\frac{d^2y_1}{dx^2}+2x\frac{dy_1}{dx}-my_1=0, \quad y_1=\frac{dy}{dx},$$

$$(1+x^2)\frac{d^2y_2}{dx^2}+4x\frac{dy_2}{dx}-(m-2)\,y_2=0, \quad y_2=\frac{dy_1}{dx}; \text{ \&c.}$$

As a third example, suppose

$$(1-x^2)\frac{d^2y}{dx^2}+mx\frac{dy}{dx}-ry=0, \quad m=p+q-1, \quad r=pq.$$

The scale is $n(n-1)\,a_n=\{(n-2)(n-3)-m(n-2)+r\}\,a_{n-2}$,
or $n(n-1)\,a_n=(n-p-2)(n-q-2)\,a_{n-2}$.

This gives

$$y=a_0\left\{1+\frac{pq}{2}x^2+\frac{p(p-2)\,q\,(q-2)}{2\,.\,3\,.\,4}x^4+\text{\&c.}\right\}$$

$$+\,a_1\left\{x+\frac{(p-1)(q-1)}{2\,.\,3}x^3+\frac{(p-1)(p-3)(q-1)(q-3)}{2\,.\,3\,.\,4\,.\,5}x^5+\text{\&c.}\right\}$$

If one of the quantities p, q, be an even and the other an odd integer, both these series will terminate, and we shall have the complete integral in finite terms. Let $p=2$, $q=1$; $y=C(1+x^2)+C'x$ is the complete integral of

$$(1-x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y=0.$$

If one only of the quantities p, q be an integer, even or odd, we shall have a particular integral in finite terms; and we can express the other particular integral in finite terms by means of ordinary integrals.

Differentiating the proposed, making $y_1=\dfrac{dy}{dx}$, we have

$$(1-x^2)\frac{d^2y_1}{dx^2}+m'x\frac{dy_1}{dx}-r'y=0,$$

$$m'=m-2=p+q-3, \quad r'=r-m=(p-1)(q-1).$$

Hence p and q, by this operation, are each diminished of a unit. If therefore either of them be integer, we can, by repeating the operation, take away the last term from the equation, and render it integrable; but we should obtain only a particular integral.

For a fourth example, let

$$(1+x^2)\frac{d^2y}{dx^2}-mx\frac{dy}{dx}+ry=0,$$

m and r as before.

The scale of this is

$$n(n-1)a_n + (n-p-2)(n-q-2)a_{n-2} = 0;$$

whence we find

$$y = a_0\left\{1 - \frac{pq}{2}x^2 + \frac{p(p-2)q(q-2)}{2\,.\,3\,.\,4}x^4 - \&c.\right\}$$

$$+ a_1\left\{x - \frac{(p-1)(q-1)}{2\,.\,3}x^3 + \frac{(p-1)(p-3)(q-1)(q-3)}{2\,.\,3\,.\,4\,.\,5}x^5 - \&c.\right\}$$

If p and q be an even and an odd integer, both these series terminate. Let $p = 2$, $q = 1$, and we have $y = C(1-x^2) + C'x$ for the complete integral of

$$(1+x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

If one only of the quantities p, q be integer, we find only a particular integral in finite terms.

Differentiating the proposed once, making $\frac{dy}{dx} = y_1$, we obtain a similar equation, p and q being each diminished by unity. By successive differentiation it integrates itself, as in the last example, p or q being an integer.

Having obtained one particular integral, the other may be sometimes found by a simple transformation, as in the two next examples.

Example fifth.

$$\text{Let } x^2\frac{d^2y}{dx^2} - m\frac{dy}{dx} - ry = 0, \quad r = p(p-1).$$

The scale $mna_n = \{(n-1)(n-2) - r\}a_{n-1}$, gives immediately

$$y = a_0\left\{1 - \frac{r}{m}x + \frac{r^2}{2m^2}x^2 - \frac{r^2(r-1\,.\,2)}{2\,.\,3m^3}x^3 + \frac{r^2(r-1\,.\,2)(r-2\,.\,3)}{2\,.\,3\,.\,4m^4}x^4 - \&c.\right\}$$

This is a particular integral, and it terminates in the case supposed. If v denote this integral, we find $y = Cv\int\frac{dx\,c^{-\frac{m}{x}}}{v^2}$ for the other particular integral, which we might suppose could not be freed from the sign of integration. But if we make $y = zc^{-\frac{m}{x}}$, the proposed becomes

$$x^2\frac{d^2z}{dx^2} + m\frac{dz}{dx} - \left(r + \frac{2m}{x}\right)z = 0.$$

The scale is $n(n-1)a_{n+1}+\{n(n-1)-r\}a_n=0,$

or $n(n-1)a_{n+1}+(n-p)(n+p-1)a_n=0,$

which gives

$$z=a_2\left\{x^2+\frac{(p+1)(p-2)}{m}x^3+\frac{(p+1)(p+2)(p-2)(p-3)}{2m^2}x^4+\&c.\right\}$$

This terminates; and it is obvious that it is not the particular integral before found, since it cannot by any means be reduced to it: we have therefore the complete integral of the proposed in finite terms. And multiplying it by $c^{\frac{m}{x}}$, we have that of the equation

$$x^2\frac{d^2z}{dx^2}+m\frac{dz}{dx}-\left(r+\frac{2m}{x}\right)z=0.$$

We can easily deduce also the integrals of the following:

$$x^2\frac{d^2y_1}{dx^2}+(2x-m)\frac{dy_1}{dx}-ry_1=0, \qquad y_1=\frac{dy}{dx},$$

$$x^2\frac{d^2y_2}{dx^2}+(4x-m)\frac{dy_2}{dx}-(r-2)y_2=0, \qquad y_2=\frac{dy_1}{dx},$$

$$x^2\frac{d^2y_3}{dx^2}+(6x-m)\frac{dy_3}{dx}-(r-2.3)y_3=0, \quad y_3=\frac{dy_2}{dx}, \ \&c.$$

$$x^2\frac{d^2y_1}{dx^2}-(2x+m)\frac{dy_1}{dx}-(r-2)y_1=0, \qquad y_1=\int y\,dx,$$

$$x^2\frac{d^2y_2}{dx^2}-(4x+m)\frac{dy_2}{dx}-(r-2.3)y_2=0, \quad y_2=\int y_1\,dx, \ \&c.$$

Continuing these processes, the last term will ultimately vanish, and the equation become integrable.

Example sixth.

Let $\dfrac{d^2y}{dx^2}+q\dfrac{dy}{dx}=\dfrac{m}{x^2}y, \quad m=p(p-1).$

Here $\{m-n(n-1)\}a_n=(n-1)qa_{n-1}.$

Hence we find

$$y=a_0\left\{1-\frac{m}{qx}+\frac{m(m-1.2)}{2q^2x^2}-\frac{m(m-1.2)(m-2.3)}{2.3\,q^3x^3}+\&c.\right\}$$

a particular integral, which terminates.

Make $y=zc^{-qx}$, and we have the transformed

$$\frac{d^2z}{dx^2}-q\frac{dz}{dx}=\frac{m}{x^2}z.$$

Hence we have only to change q into $-q$ in the value of y just found, and we have

$$z = a_0 \left\{1 + \frac{m}{qx} + \frac{m(m-1.2)}{2q^2x^2} + \frac{m(m-1.2)(m-2.3)}{2.3\,q^3x^3} + \&c.\right\}$$

therefore

$$y = C\left\{1 - \frac{m}{qx} + \frac{m(m-1.2)}{2q^2x^2} - \&c.\right\} + C'c^{-qx}\left\{1 + \frac{m}{qx} + \frac{m(m-1.2)}{2q^2x^2} + \&c.\right\}$$

is the complete integral of the proposed.

If in the value of y we change q into $-q$, we have the complete integral of

$$\frac{d^2y}{dx^2} - q\frac{dy}{dx} = \frac{m}{x^2}y.$$

Make $y = x^p u$, and we find by substitution

$$x\frac{d^2u}{dx^2} + (2p + qx)\frac{du}{dx} + qpu = 0.$$

The integral of this last, therefore, is immediately derivable from that of the proposed equation.

Again, make $\dfrac{dy}{dx} + qy = w$, and we have

$$x^2\frac{dw}{dx} = my, \quad x^2\frac{d^2w}{dx^2} + 2x\frac{dw}{dx} = m\frac{dy}{dx}.$$

Add to this the preceding multiplied by q, and we get

$$x^2\frac{d^2w}{dx^2} + (2x + qx^2)\frac{dw}{dx} = mw.$$

We have therefore the integral of this last also.

Make $y = vc^{-\frac{1}{2}qx}$, and the proposed becomes

$$\frac{d^2v}{dx^2} - \tfrac{1}{4}q^2v = \frac{m}{x^2}v;$$

of which we have also the integral. And changing q into $2q\sqrt{-1}$ in the integral of this last, we have that of

$$\frac{d^2v}{dx^2} + q^2v = \frac{m}{x^2}v.$$

But the integrals of the two last are better found directly; and the operation will show us, that when we cannot immediately obtain one particular integral, we may sometimes by a simple transformation find them both.

$$\text{Let } \frac{d^2y}{dx^2} - q^2y = \frac{m}{x^2}y, \quad m = p(p-1).$$

We cannot here obtain an integral immediately.

Make $y = zc^{-qx}$, then $\dfrac{d^2z}{dx^2} - 2q\dfrac{dz}{dx} = \dfrac{m}{x^2}z.$

The scale of this is

$$\{n(n-1) - m\}\, a_n = (n-1)\, 2qa_{n-1};$$

which gives

$$z = a_0\left\{1 + \frac{m}{2qx} + \frac{m(m-1\,.\,2)}{2(2qx)^2} + \&c.\right\}$$

$$\text{and } y = Cc^{-qx}\left\{1 + \frac{m}{2qx} + \frac{m(m-1\,.\,2)}{2(2qx)^2} + \&c.\right\}$$

If we change q into $-q$, we have

$$y = C'c^{qx}\left\{1 - \frac{m}{2qx} + \frac{m(m-1\,.\,2)}{2(2qx)^2} - \&c.\right\}$$

These are two particular integrals of the proposed, and give for the complete integral

$$y = Cc^{-qx}\left\{1 + \frac{m}{2qx} + \frac{m(m-1.2)}{2(2qx)^2} + \&c.\right\} + C'c^{qx}\left\{1 - \frac{m}{2qx} + \frac{m(m-1.2)}{2(2qx)^2} - \&c.\right\}$$

Changing q into $q\sqrt{-1}$, and suitably changing the arbitraries C, C', we shall easily find, by taking the odd and even terms separately,

$$y = C\sin(qx+\beta)\,.\left\{1 - \frac{m(m-1.2)}{1.2\,(2qx)^2} + \frac{m(m-1.2)(m-2.3)(m-3.4)}{1.2.3.4\,(2qx)^4} - \&c.\right\}$$

$$+\ C'\cos(qx+\beta)\,.\left\{\frac{m}{2qx} - \frac{m(m-1\,.\,2)(m-2\,.\,3)}{1\,.\,2\,.\,3\,(2qx)^3} + \&c.\right\}$$

for the complete integral of

$$\frac{d^2y}{dx^2} + q^2y = \frac{m}{x^2}y.$$

We might have ascending series instead of descending ones; but it would only be beginning at the other end of the series. And it may be observed, that the method employed in the last example is one that has been long applied to the integration of kindred equations.

In the last equation make $y = x^r z$, and it becomes

$$\frac{d^2z}{dx^2} + \frac{2r}{x}\frac{dz}{dx} + q^2z = 0, \quad r(r-1) = m.$$

Again, make $y = x^{-r}z$, and we have

$$\frac{d^2z}{dx^2} - \frac{2r}{x}\frac{dz}{dx} + q^2z = 0, \quad r(r+1) = m.$$

Thus the integrals of these two last are immediately obtained from the preceding value of y.

When we cannot obtain a complete integral, it will sometimes happen that a particular integral will suffice: and sometimes by means of ordinary integrals we can, with a little tact, obtain another in a simple form.

Let $\dfrac{d^2y}{dx^2} + q^2x\dfrac{dy}{dx} + q^2my = 0$, m an affirmative integer.

$$\text{Make } y = zc^{-\frac{1}{2}q^2x^2}.$$

$$\text{This gives } \frac{d^2z}{dx^2} - q^2x\frac{dz}{dx} + q^2mz = 0,$$

of which the scale is $n(n-1)\,a_n = (n-m-2)\,q^2a_{n-2}$.

Hence we find

$$z = C\left\{1 - \frac{m}{2}q^2x^2 + \frac{m(m-2)}{2.3.4}q^4x^4 - \frac{m(m-2)(m-4)}{2.3.4.5.6}q^6x^6 + \&c.\right\}$$

$$+ C'\left\{qx - \frac{m-1}{2.3}q^3x^3 + \frac{(m-1)(m-3)}{2.3.4.5}q^5x^5 - \&c.\right\}$$

and therefore

$$y = Cc^{-\frac{1}{2}q^2x^2}\left\{1 - \frac{m}{2}q^2x^2 + \frac{m(m-2)}{2.3.4}q^4x^4 - \&c.\right\} + C'c^{-\frac{1}{2}q^2x^2}\left\{qx - \frac{m-1}{2.3}q^3x^3 + \&c.\right\}$$

In each of these integrals one of the series which they contain will terminate; the first if m be even, the second if odd.

If we change q^2 into $-q^2$, and suitably change the arbitraries, we have

$$\frac{d^2y}{dx^2} - q^2x\frac{dy}{dx} - q^2my = 0, \quad \frac{d^2z}{dx^2} + q^2x\frac{dz}{dx} - q^2mz = 0,$$

$$z = C\left\{1 + \frac{m}{2}q^2x^2 + \frac{m(m-2)}{2.3.4}q^4x^4 + \&c.\right\} + C'\left\{qx + \frac{m-1}{2.3}q^3x^3 + \&c.\right\}$$

$$y = Cc^{\frac{1}{2}q^2x^2}\left\{1 + \frac{m}{2}q^2x^2 + \frac{m(m-2)}{2.3.4}q^4x^4 + \&c.\right\} + C'c^{\frac{1}{2}q^2x^2}\left\{qx + \frac{m-1}{2.3}q^3x^3 + \&c.\right\}$$

Thus we have, in finite terms, a particular integral of each of the four preceding equations. Before we proceed to find the other particular integral, it will be convenient to have that already found in a descending series.

If $\dfrac{d^2y}{dx^2} + q^2x\dfrac{dy}{dx} - q^2my$; $n(n-1)\,a_n = (m-n+2)\,q^2a_{n-2}$, and

$$y = Cx^m \left\{1 + \frac{m(m-1)}{2q^2x^2} + \frac{m(m-1)(m-2)(m-3)}{2 \cdot 4q^4x^4} + \text{\&c.}\right\},$$

which terminates whether m be odd or even.

Changing q^2 into $-q^2$, we have

$$\frac{d^2y}{dx^2} - q^2x\frac{dy}{dx} + q^2my = 0,$$

and $y = Cx^m \left\{1 - \frac{m(m-1)}{2q^2x^2} + \frac{m(m-1)(m-2)(m-3)}{2 \cdot 4q^4x^4} - \text{\&c.}\right\}$:

and from what has been done,

$$\frac{d^2y}{dx^2} + q^2x\frac{dy}{dx} + q^2my = 0;$$

$$y = Cx^m c^{-\frac{1}{2}q^2x^2} \left\{1 - \frac{m(m-1)}{2q^2x^2} + \frac{m(m-1)(m-2)(m-3)}{2 \cdot 4q^4x^4} - \text{\&c.}\right\}:$$

$$\frac{d^2y}{dx^2} - q^2x\frac{dy}{dx} - q^2my = 0;$$

$$y = Cx^m c^{\frac{1}{2}q^2x^2} \left\{1 + \frac{m(m-1)}{2q^2x^2} + \frac{m(m-1)(m-2)(m-3)}{2 \cdot 4q^4x^4} + \text{\&c.}\right\}.$$

We are now prepared to find the other particular integral. For this purpose we will take

$$\frac{d^2z}{dx^2} - q^2x\frac{dz}{dx} + q^2mz = 0.$$

$$\text{Make } z = uc^{\frac{1}{2}q^2x^2} + v\int dx\, c^{\frac{1}{2}q^2x^2},$$

where v is the particular integral already found. Substituting this value, we have

$$\frac{d^2u}{dx^2} + q^2x\frac{du}{dx} + q^2mu + 2\frac{dv}{dx} = 0.$$

If $u = \Sigma a_n x^n$, $v = \Sigma b_n x^n$, the scale is

$$n(n-1)a_n + (n+m-2)q^2a_{n-2} + 2(n-1)b_{n-1} = 0,$$
$$n(n-1)b_n - (n-m-2)q^2b_{n-2} = 0,$$

Now b_{m+2}, b_{m+4}, &c. $= 0$. Make $a_{m+1} = 0$; then a_{m+3}, a_{m+5}, &c. $= 0$, and we have

$$a_{m-1} = -\frac{2m}{2m-1} \cdot \frac{b_m}{q^2}, \quad a_{m-3} = -\frac{m(m-1)(m-2)}{2m-1} \cdot \frac{b_m}{q^4}, \text{ \&c.}$$

The scale shows that the series terminates either at a_1 or a_0, according as m is an odd or an even number. And thus we shall have

$$u = a_{m-1}x^{m-1} + a_{m-3}x^{m-3} + \ldots + a_1x, \text{ or } a_0.$$

Multiplying the particular integral thus found by $c^{-\frac{1}{2}q^2x^2}$, and then changing q^2 into $-q^2$, we shall have the other three particular integrals.

By successive differentiation and integration of the four equations last integrated, we should ultimately make the last term to vanish, and the equations to integrate themselves. But for the most part we obtain only particular integrals by such a process.

As another example, let $\frac{d^2y}{dx^2} + q\frac{dy}{dx} + \frac{mq}{x}y = 0$, m a positive integer. Make $y = zc^{-qx}$, and we have the transformed

$$\frac{d^2z}{dx^2} - q\frac{dz}{dx} + \frac{mq}{x}z = 0.$$

$$\text{Here } n(n-1)a_n = (n-m-1)qa_{n-1};$$

$$\text{whence } z = Cx\left\{1 - \frac{m-1}{2}qx + \frac{(m-1)(m-2)}{2^2 \,.\, 3}q^2x^2 - \&\text{c.}\right\}$$

$$y = Cxc^{-qx}\left\{1 - \frac{m-1}{2}qx + \frac{(m-1)(m-2)}{2^2 \,.\, 3}q^2x^2 - \&\text{c.}\right\}$$

Changing q into $-q$, we obtain particular integrals of

$$\frac{d^2y}{dx^2} - q\frac{dy}{dx} - \frac{mq}{x}y = 0, \quad \frac{d^2z}{dx^2} + q\frac{dz}{dx} - \frac{mq}{x}z = 0.$$

All these integrals terminate in the case supposed. Found by a descending series, they are

$$z = Cx^m\left\{1 - \frac{m(m-1)}{qx} + \frac{m(m-1)^2(m-2)}{2q^2x^2} - \frac{m(m-1)^2(m-2)^2(m-3)}{2 \,.\, 3\, q^3x^3} + \&\text{c.}\right\}$$

$$y = Cx^mc^{-qx}\left\{1 - \frac{m(m-1)}{qx} + \frac{m(m-1)^2(m-2)}{2q^2x^2} - \&\text{c.}\right\}$$

Changing q into $-q$, we obtain the corresponding integrals of the two last equations.

In order to have the other particular integral, we will take the equation

$$\frac{d^2z}{dx^2} - q\frac{dz}{dx} + \frac{qm}{x}z = 0,$$

and we make

$$z = uc^{qx} + v\int\frac{dx}{x}c^{qx},$$

where v is the particular integral which we have found, whether by an ascending or a descending series. By substitution we find

$$\frac{d^2u}{dx^2} + q\frac{du}{dx} + \frac{qm}{x}u + \frac{2}{x}\frac{dv}{dx} - \frac{1}{x^2}v = 0.$$

If $u = \Sigma a_n x^n$, $v = \Sigma b_n x^n$, the scale is

$$n(n-1)a_n + (n+m-1)qa_{n-1} + (2n-1)b_n = 0,$$

$$\text{and } n(n-1)b_n = (n-m-1)qb_{n-1}.$$

As we only want a particular integral, we have one of the quantities a_0, a_1, a_2, &c. more than we want, since we have an arbitrary in the value of v. We may therefore make $a_m = 0$; then a_{m+1}, a_{m+2}, ... $= 0$; since $b_{m+1} = 0$, $b_{m+2} = 0$, &c.

We begin at the further extremity of the series, and find the quantities a_0, a_1, &c. in a retrograde order, and we may take either the ascending or descending series for v; but the latter is more convenient. Thus we have

$$a_{m-1} = -\frac{b_m}{q} = -\frac{C}{q}, \quad a_{m-2} = \{m(m-1)-1\}\frac{C}{q^2}, \text{ \&c.;}$$

$$\text{also } a_0 = -\frac{b_1}{mq}, \quad a_{-1} = 0;$$

$$\text{therefore } u = a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \ldots + a_0.$$

From this, by multiplying by c^{qx}, and changing q into $-q$ as before, we derive the other three particular integrals.

It is often very important to attend particularly to the scale; for it will often direct us to the right mode of finding the integral, when otherwise we might have overlooked it. We will show this in a few examples, though it must in a great measure have been apparent as we have proceeded.

$$\text{Let } \frac{d^2y}{dx^2} + \frac{m}{x}\frac{dy}{dx} + \frac{r}{x^2}y = 0.$$

The scale is $\{n(n-1) + mn + r\}a_n = 0$, or $n(n-1) + mn + r = 0$.

Let n_1, n_2, be the values of n which satisfy this equation; then

$$y = Cx^{n_1} + C'x^{n_2}.$$

It is true the equation here taken as an example is one which we already knew how to integrate, and one of the easiest to integrate, except where the coefficients are constant: but it is not integrated so easily in any other way.

$$\text{Next, let } \frac{d^2y}{dx^2} + q^2x\frac{dy}{dx} + q^2y = 0.$$

The scale of this is

$$n(n-1)a_n + (n-1)q^2 a_{n-2} = 0;$$

which reduces to $na_n + q^2 a_{n-2} = 0$.

This last is the scale of $\dfrac{dy}{dx} + q^2 xy = 0$; which gives $y = Cc^{-\frac{1}{2}q^2x^2}$, a particular integral of the proposed. The complete integral is

$$y = Cc^{-\frac{1}{2}q^2x^2} \int dx c^{\frac{1}{2}q^2x^2} + C'c^{-\frac{1}{2}q^2x^2}.$$

The proposed may be put under the form $\dfrac{d^2y}{dx^2} + q^2 \dfrac{d(xy)}{dx} = 0$, and is immediately integrable; but we might have overlooked that circumstance if the scale had not pointed it out to us.

$$\text{Again, let } (1-x^2)\frac{d^2y}{dx^2} + 2y = 0.$$

The scale of this is

$$n(n-1)a_n = \{(n-2)(n-3) - 2\} a_{n-2} = (n-4)(n-1)a_{n-2},$$

$$\text{or } na_n = (n-4)a_{n-2}.$$

This is the scale of $(1-x^2)\dfrac{dy}{dx} + 2xy = 0$; which gives $y = C(1-x^2)$, a particular integral of the proposed. In fact, the proposed is immediately integrable, and gives

$$(1-x^2)\frac{dy}{dx} + 2xy = C.$$

These may be thought to be very simple examples, the integrability of which was easily discovered. But were I not afraid of extending this paper to too great a length, it would not be difficult to show the importance of consulting the scale in cases much less obvious.

VI.—ON THE MOTION OF A SPHERE PROJECTED ALONG A CYLINDER REVOLVING UNIFORMLY IN A VERTICAL PLANE.

By JAMES BOOTH, M.A., Principal of and Professor of Mathematics in Bristol College.

LET a hollow cylinder be supposed to revolve uniformly in a vertical plane round a horizontal axe, and let a sphere be projected along the cylinder from a given point in it

with a given velocity; to determine the motion of the sphere, excluding the effects of friction and the resistance of the atmosphere.

Let r be the distance of the sphere from the centre of motion at the end of the time t; let T be the period of the revolution of the cylinder, α the angle which the axis of the cylinder makes with the horizontal axe in its own plane at the beginning of the time t; then the forces which act on the sphere at any instant, are the force of gravity resolved along the axis of the cylinder, and the centrifugal force arising from the uniform rotation of the cylinder.

Now the angle which the axis of the cylinder makes with the horizontal axe at the end of the time t, is $\left(\alpha + 2\pi \dfrac{t}{T}\right)$; hence the force of gravity resolved along the axis of the cylinder is $-g \sin\left(\alpha + \dfrac{2\pi t}{T}\right)$, g being taken with a negative sign, as it acts towards the centre at the commencement of the motion, and the centrifugal force is $\dfrac{4\pi^2 r}{T^2}$; hence the differential equation of the motion of the sphere is

$$\frac{d^2 r}{dt^2} = \frac{4\pi^2 r}{T^2} - g \sin\left(\alpha + 2\pi \frac{t}{T}\right) \ldots\ldots (1).$$

Let $\dfrac{2\pi}{T} = k$, and equation (1) becomes

$$\frac{d^2 r}{dt^2} = k^2 r - g \sin(\alpha + kt) \ldots\ldots\ldots (2).$$

Now, to integrate this equation, using the method of the variation of parameters, let us assume

$$r = Ae^{kt} + Be^{-kt} \ldots\ldots\ldots (3),$$

which is the integral of the differential equation $\dfrac{d^2 r}{dt^2} = k^2 r$; A and B being functions of t to be determined.

Differentiating (3), we obtain

$$\frac{dr}{dt} = k(Ae^{kt} - Be^{-kt}) + \frac{dA}{dt} e^{kt} + \frac{dB}{dt} e^{-kt} \ldots\ldots (4).$$

As a first condition, let

$$\frac{dA}{dt} e^{kt} + \frac{dB}{dt} e^{-kt} = 0 \ldots\ldots (5).$$

Introducing the condition (5) into (4), and differentiating it again, we find

$$\frac{d^2r}{dt^2} = k^2 (Ae^{kt} + Be^{-kt}) + k\left(\frac{dA}{dt}e^{kt} - \frac{dB}{dt}e^{-kt}\right).$$

Eliminating from this equation the quantity $Ae^{kt} + Be^{-kt}$ by (3), we get

$$\frac{d^2r}{dt^2} = k^2r + k\left(\frac{dA}{dt}e^{kt} - \frac{dB}{dt}e^{-kt}\right) \quad \ldots\ldots (6).$$

Comparing this equation with (2), we find

$$\frac{dA}{dt}e^{kt} - \frac{dB}{dt}e^{-kt} = \frac{-g}{k}\sin(a + kt) \ldots\ldots (7).$$

Eliminating successively $\frac{dB}{dt}$, $\frac{dA}{dt}$, from equations (5) and (7), and integrating by parts, we obtain

$$\left.\begin{aligned} A &= C + \frac{g}{4k^2}e^{-kt}\{\sin(a+kt) + \cos(a+kt)\} \\ B &= C' + \frac{g}{4k^2}e^{kt}\{\sin(a+kt) - \cos(a+kt)\} \end{aligned}\right\} \ldots . (8),$$

C and C' being arbitrary constants. Substituting these values of A and B in (3), we find

$$r = Ce^{kt} + C'e^{-kt} + \frac{g}{2k^2}\sin(a + kt) \ldots\ldots (9).$$

To determine the values of C and C'.

Let R be the initial distance, and V the velocity of projection; then R and V are the values of r and $\frac{dr}{dt}$ when $t = 0$; hence

$$\left.\begin{aligned} R &= C + C' + \frac{g}{2k^2}\sin a \\ \frac{V}{k} &= C - C' + \frac{g}{2k^2}\cos a \end{aligned}\right\} \ldots . (10).$$

From these equations, determining the values of C and C', and substituting them in (9), we obtain finally the equation

$$r = \left(\frac{R}{2} - \frac{g}{4k^2}\sin a\right)(e^{kt} + e^{-kt})$$
$$+ \left(\frac{V}{2k} - \frac{g}{4k^2}\cos a\right)(e^{kt} - e^{-kt}) + \frac{g}{2k^2}\sin(a + kt) \ldots . (11),$$

from which we obtain the value of r in terms of the time.

1. Let the initial distance $= \frac{g}{2k^2} \sin \alpha$, and the initial velocity $= \frac{g}{2k} \cos \alpha$; then equation (11) is changed into

$$r = \frac{g}{2k^2} \sin (\alpha + kt) \ldots\ldots (12).$$

the equation of a circle, whose lowest point is in the centre of motion, and whose radius $= \frac{g}{4k^2}$.

In order to simplify, let the initial position of the cylinder be vertical, and the initial velocity $= 0$; then $\alpha = \frac{\pi}{2}$, $V = 0$, and

$$r = \left(\frac{R}{2} - \frac{g}{4k^2}\right)(e^{kt} + e^{-kt}) + \frac{g}{2k^2} \cos kt \ldots\ldots (13).$$

In the first place, if the initial distance be taken $= \frac{g}{2k^2}$, the orbit which the sphere describes is a vertical circle, touching at its lowest point the horizontal line passing through the centre of motion; and this circle is described by the sphere in a semi-revolution of the cylinder.

At the end of one or any odd number of revolutions, the sphere will be found at the negative side of the origin in the cylinder, at the distance R from the centre; while at the end of two or any even number of revolutions it returns to the same point of space and the same position in the cylinder.

2. When R is $\gtrless \frac{g}{2k^2}$, let $R - \frac{g}{2k^2} = 2\delta$, and (13) becomes

$$r = \delta (e^{kt} + e^{-kt}) + \frac{g}{2k^2} \cos kt \ldots\ldots (14).$$

In this case the sphere will alternately pass to and fro through the centre of motion so long as $\delta (e^{kt} + e^{-kt})$ is less than $\frac{g}{2k^2}$, making excursions in the negative arm of the cylinder, constantly diminishing, until at length it remains altogether on the positive side of the origin, receding with a constantly accelerated velocity from the centre of motion, ever approaching in the upper part of the orbit to the centre by a fixed quantity, and receding from it in the lower part by the same.

The sphere in this case undulates along a curve which is asymptotic to a logarithmic spiral, the angle between whose

tangent and rad. vec. is equal to half a right angle, and whose parameter is δ.

3. Let the sphere have no initial velocity, and let the initial position of the cylinder be horizontal; then $V = 0$, $a = 0$; putting these values in (11),

$$r = \left(\frac{R}{2} - \frac{g}{4k^2}\right) e^{kt} + \left(\frac{R}{2} + \frac{g}{4k^2}\right) e^{-kt} + \frac{g}{2k^2} \sin kt.$$

Let $R = \frac{g}{2k^2}$, then

$$r = \frac{g}{2k^2} (e^{-kt} + \sin kt);$$

in which case the orbit approaches indefinitely to a circle.

When the motion of the cylinder is very slow, k becomes very small, and (11) is changed into $r = R + \frac{R}{2} k^2t^2 - \frac{g}{2} t^2$; which, when $k = 0$, gives $r = R - \frac{gt^2}{2}$, as it ought to be.

Let the period of revolution be $2''$; then $T = 2''$,

$$k = \frac{2\pi}{T} = \pi, \quad g = 32.1908 \text{ feet};$$

$$\text{hence } R = \frac{g}{2k^2} = \frac{g}{2\pi^2} = 1.630 \text{ feet},$$

the diameter of the vertical circle which the sphere describes.

VII.—ON THE DETERMINATION OF THE INTENSITY OF VIBRATION OF WAVELETS DIVERGING FROM EVERY POINT OF A PLANE WAVE.

Calculations of phenomena of diffraction on the principles of Fresnel, are made by supposing that from every part of the front of a primary wave, in any position, small waves diverge, proportional in intensity to the superficial extent of the part, and diminishing in intensity proportionally to the distance through which they diverge. So that if $a \,.\, \sin \frac{2\pi}{\lambda}(vt - x)$ re-

present the disturbance of a particle in the front of the primary wave, from every element dS of the surface we may suppose a small wave to proceed, and the disturbance of a particle by the small wave to be represented by

$$\frac{b}{r} . \, dx \, dy \sin \frac{2\pi}{\lambda} . (vt - c - r).$$

The value of the coefficient b depends on that of a, but the relation is not, I believe, given by Fresnel nor by Airy, nor, as far as I know, by any other writer. I propose to determine it from the principle, that if the primary wave be plane, the disturbance caused by the small waves at any point in front of the primary wave must be the same as the disturbance which would be caused by the primary wave itself. Let c be the length of a perpendicular drawn from a point A to the front of a plane wave, r and u the distances of any element of the wave from the point A, and from the intersection of the perpendicular with the front of the primary wave, so that $r^2 = c^2 + u^2$. Supposing the front of the wave to be divided into elementary zones by concentric circles of radii u and $u + du$, the area of each is $2\pi u \, du = 2\pi r \, dr$. The disturbance produced at the point A by waves proceeding from this zone will be, since the waves proceeding from each point are in the same phase,

$$b \, . \, 2\pi \, dr \sin \frac{2\pi}{\lambda} . (vt - r),$$

and the total disturbance

$$= 2\pi b \int_c^\infty \sin \frac{2\pi}{\lambda} (vt - r) \, dr$$

$$= b\lambda \left\{ \cos \frac{2\pi}{\lambda} (vt - \infty) - \cos \frac{2\pi}{\lambda} (vt - c) \right\},$$

$$= b\lambda \sin \left\{ \frac{2\pi}{\lambda} . (vt - c) - \frac{\pi}{2} \right\},$$

(since the sines and cosines of infinite arcs are 0,)

$$= b\lambda \sin \frac{2\pi}{\lambda} \left\{ vt - \left(c + \frac{\lambda}{4} \right) \right\},$$

so that $b\lambda = a$, and therefore the coefficients of the small waves diverging from dS must be $\dfrac{a dS}{r\lambda}$.

It would also appear from this, that we should consider the phase of the small wave to be expressed by

$$\sin \frac{2\pi}{\lambda} \left\{ vt - \left(r - \frac{\lambda}{4} \right) \right\};$$

so that if the substitution of small waves for the primary wave were real instead of hypothetical, there would be a loss of a quarter of an undulation.

H. T.

VIII.—MATHEMATICAL NOTE.

A Suggestion in Notation.—Of all the repetitions which want of notation compels, that of n, $n\,\frac{n-1}{2}$, $n\,\frac{n-1}{2}\,\frac{n-2}{3}$, is one of the worst. If these were to be called n_1, n_2, n_3, &c. the notation could not be permanent, since n_1, n_2, &c. are used for any set of quantities following a law. But if 1_n, 2_n, 3_n, &c. be used, no existing notation will be interfered with, except in the *general term* k_n, which it is seldom wanted to pass over quickly and frequently.

These symbols, 1_n, 2_n, 3_n, &c. might be read '1 out of n', '2 out of n', &c. or '1 of n', '2 of n', &c. in abbreviation of 'the number of ways in which 1, 2, &c. may be selected out of n'. The following are some instances of their use:

$$(1+x)^n = 1 + 1_n x + 2_n x^2 + 3_n x^3 + 4_n x^3 + \ldots.$$

$$k_{m+n} = k_m + (k-1)_m 1_n + (k-2)_m 2_n + (k-3)_m 3_n + \ldots.$$

$$\frac{k_n}{l_n} = \frac{(k-l)_{n-l}}{(k-l)_k}, \quad k_n = \frac{\Gamma(n)}{\Gamma(k)\,\Gamma(n-k)}, \quad (k+1)_n = k_n \times \frac{n-k}{k+1}.$$

A. D. M.

ERRATUM.

Page 22, line 4, *for* a *read* $\frac{a_{m-2}}{a_{m-1}}$.